AUTOMORPHISMS AND RANGE FAMILIES OF TRANSFORMATION SEMIGROUPS

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ABSTRACT

The problem of describing all automorphisms of a given semigroup of transformations of a set X has interested a number of mathematicians in the past fifty years. J. Schreier showed that all automorphisms of the full transformation semigroup $\mathcal{T}_X$ are *inner*, and A. Mal'cev showed that the same property holds for any ideal of $\mathcal{T}_X$. More recently J. Symons showed that all automorphisms of any $\mathcal{G}_X$-normal semigroup over a *finite* set X are inner, while B.M. Schein produced the same result for $\mathcal{G}_X$-normal semigroups of 1-1 transformations over an *infinite* set X.

Chapters 2 and 3 of this thesis constitute a contribution toward the solution of the problem of describing all automorphisms of a given semigroup of transformation of an *infinite* set X. In Chapter 2 we extend the well-known result from group theory, namely that any normal group of bijections of an infinite set X has only inner automorphisms, to an analogous one in semigroup theory. We show that any $\mathcal{G}_X$-normal semigroup of transformations of an infinite set X has only inner automorphisms. In Chapter 3 (which is a joint work with K.C. O'Meara and G.R. Wood) we give the description of all automorphisms of an arbitrary *Croisot-Teissier* semigroup. They offer a rich variety, from inner to "locally" inner, to thoroughly outer. We also present a description of Green's relations on Croisot-Teissier semigroups.

In Chapter 4 we define a *normal* subset of the power set $\mathcal{P}_X$ of an infinite set X. We characterize all normal subsets of $\mathcal{P}_X$ which serve as sets of ranges of semigroups of total transformations of X.

In Chapter 5 for a particular normal subset of $\mathcal{P}_X$ we give necessary and sufficient conditions for an order-automorphism to be determined by a bijection of X (that is, *induced*). We then characterize those normal subsets of $\mathcal{P}_X$ for which all order-automorphisms are induced.

NOTATION

In this thesis we adopt the following system of numbering: theorems, propositions, notation, remarks, definitions and examples are numbered by the Chapter and Section in which they appear and by their occurrence in the Section. Thus Proposition 1.2.3 (or just 1.2.3) refers to the third numbered statement (which happened to be a proposition) in the second section of Chapter 1. A similar numbering is used for displayed formulae. These do not run concurrently with the theorems but have their own numbering. We distinguish the numbering of formulae from theorems by parentheses. Thus "see (1.2.3)" means "see formula (1.2.3)".

We include below a short list of symbols together with the numbers of pages where these symbols first occurred. This list is by no means exhaustive, but does contain the most frequently used items.

Notation		*Page*
A'	the complement of the set A	8
A_α	$\bigcup_{M_\alpha} A$	60
A^c	the σ-closure of the set A by an equivalence σ	42
$\mathcal{A}_\alpha$	$\bigcup_\beta \mathcal{A}_{\alpha,\beta}$	130,142
$\mathcal{A}_{\alpha,\beta}$	$\{A \in \mathcal{A} \subseteq \mathcal{P}_X \mid \lvert A\rvert = \alpha,\ \lvert A'\rvert = \beta\}$	142
$\mathcal{A}_L$	$\bigcup_\beta \mathcal{A}_{\lvert X\rvert,\beta}$	143
$\mathcal{A}_S$	$\bigcup_\alpha \mathcal{A}_{\alpha,\lvert X\rvert}$	142
$\mathcal{A}^*$	the reflection of $\mathcal{A} \subseteq \mathcal{P}_X$	143
Alt_X	the alternating group on X	4,5
Aut S	the group of all automorphisms of the semigroup S	2
α^+	the cardinal successor of $\alpha \geqslant \aleph_0$	3

$\mathcal{B}(\&_i)$	$\{E \in \&_i \mid E \subseteq X - W\}$, the set of "bad" classes of $\&_i$	94
$\mathcal{C}_r$	$\{A \mid$ for some w.s. B, $A \subseteq B$ and $\lvert B - A\rvert = r\}$	41
$\mathcal{C}_X$	the semigroup of all constant selfmaps of X	2
CT(p,r)	CT(X,&,p,r)	41
CT(X,&,p,q)	the Croisot-Teissier semigroup on X, & of type (p,q)	41
$\mathcal{D}$	Green's relation	118
D_{f_1,f_2}	$\{f_1(x), f_2(x) \mid x \in \mathcal{D}_{f_1,f_2}\}$	16
$\mathcal{D}_{f_1,f_2}$	$\{x \in X \mid f_1(x) \neq f_2(x)\}$	16
def f	$\lvert X - R(f)\rvert$, $f \in \mathcal{T}_X$	127
defect$(A,\mathcal{B})$	the defect of a $\mathcal{C}_q$ set A in $\mathcal{B} \in \&$	104
&	$\{\&_i\}_{i \in I}$	41
Φ	the empty set	10
$\mathcal{G}_X$	the group of all bijections on X	2
$\mathcal{H}$	Green's relation	118
I_α	$\{f \in CT(X,\&,p,q) \mid R(f) \in \mathcal{M}_\alpha\}$	60
ι_X	the identity relation or transformation of X	4
Inn S	the group of inner automorphisms of the semigroup S	2
$\mathcal{J}$	Green's relation	118
$\mathcal{L}$	Green's relation	118
$\mathcal{L}_A$	a point left ideal	23
$\mathcal{L}_{f_1,f_2}$	a function left ideal	24
$\mathcal{M}$	a maximal family of λ-related $\mathcal{C}_q$ sets	59
$\mathcal{N}_X$	the set of all normal subsets of $\mathcal{P}_X$	129
$\mathbf{N}$	the set of natural numbers	5
O-Aut $\mathcal{A}$	the group of order-automorphisms of $\mathcal{A} \subseteq \mathcal{P}_X$	139
$\mathcal{P}_X$	the power set of X	1
p.p.	partition preserving	44

$\pi(f)$	the partition $\{f^{-1}(x) \mid x \in R(f)\}$ of a map f	6
$\pi(S)$	$\{\pi(f) \mid f \in S\}$	13
Q	$CT(W,\mathcal{E}',p,q)$	83
$R(f)$	the range $f(X)$ of a map f	3
$R(S)$	$\{R(f) \mid f \in S\}$	13
$\mathcal{R}$	Green's relation	118
$\mathcal{R}_X$	the set of all subsets of $\mathcal{P}_X$ which serve as range families for semigroups of transformations	129
$\mathcal{R}_x$	a point right ideal	15
$\mathcal{R}_{f_1,f_2}$	a function right ideal	15
rank f	the rank $\lvert R(f)\rvert$ of a map f	3
ρ	$\bigcap_{i\in I} \mathcal{E}_i$	43
S	a semigroup of transformations	2
$\bar{S}$	$CT(\bar{W},\bar{\mathcal{E}},p,q)$	54
σ-def $\mathcal{A}$	$\{\lvert A'\rvert \mid A \in \mathcal{A} \subseteq \mathcal{P}_X\}$	128
σ-rank $\mathcal{A}$	$\{\lvert A\rvert \mid A \in \mathcal{A} \subseteq \mathcal{P}_X\}$	128
σ-def S	$\{\text{def } f \mid f \in S\}$	128
σ-rank S	$\{\text{rank } f \mid f \in S\}$	128
T_i	$\{f \in CT(X,\mathcal{E},p,q) \mid \pi(f) = \mathcal{E}_i\}$	41
$\mathcal{T}_X$	the full transformation semigroup on X	2
W	the union of all w.s. set	43
w.s.	well-separated	41
$\mathbb{Z}$	the set of integers	45

CHAPTER 1

INTRODUCTION AND HISTORICAL REMARKS

1.0. INTRODUCTION

The study of total transformation semigroups on an infinite set X which we pursue in this thesis is directed towards a solution of two problems:

Problem 1. A description of all automorphisms of a given semigroup of total transformations;

Problem 2. A characterisation of all subsets of the power set P_X which serve as sets of ranges of semigroups of transformations of X.

Many authors in the past fifty years have investigated Problem 1 for different semigroups of transformations. In Section 1.1 we present a historical account of this work. This also includes a brief discussion of Chapters 2 and 3, which contain our contribution towards a solution of Problem 1. (Chapter 3 is a result of joint work with K.C. O'Meara and G.R. Wood, [10]). We also show how the study in Set Theory pursued in Chapter 5 is connected with Problem 1.

Problem 2 was suggested by B.M. Schein and to our knowledge has been solved only for the case of monogenic semigroups of partial transformations by P.M. Olonichev [17]. Although a study of semigroups of partial transformations lies beyond the scope of the present thesis we feel an obligation to present at least a brief description of Olonichev's results, since there is no English translation of [17] yet available. This together with a summary of our investigation of Problem 2 occupies Section 1.2.

1.1. HISTORY OF THE STUDY OF AUTOMORPHISMS OF TRANSFORMATION SEMIGROUPS

Let X be an arbitrary set and S be a semigroup of total transformations of X with the composition of two transformations f and g in S defined by

$$fg(x) = f(g(x)), \text{ for all } x \text{ in } X.$$

Denote by Aut S the group of all automorphisms of S. We say that $\phi \in$ Aut S is *inner* if

$$\phi(f) = hfh^{-1}, \text{ for all } f \text{ in } S,$$

where h is a fixed element of G_X, the group of all bijections of X. Let Inn S denote the group of all inner automorphisms of S. If Aut S - Inn S is non-empty then it consists of *outer* automorphisms of S.

Let T_X and C_X denote the semigroups of all transformations of X and of all constant transformations of X respectively. Certainly, C_X is a subsemigroup of T_X. In 1937 J. Schreier [21] showed that all automorphisms of T_X are inner. His proof is based on the observation that every automorphism ϕ of T_X maps constants onto constants, so that

$$\phi(C_X) = C_X .$$

This leads to a bijection h of X, which, in turn, determines ϕ such that

$$\phi(f) = hfh^{-1}, \text{ for each } f \in T_X .$$

Schreier also pointed out that this proof can be carried out for any subsemigroup S of T_X containing all constants:

$$C_X \subseteq S \subseteq T_X .$$

This allows him to deduce the following important result about automorphisms of transformation semigroups:

<u>THEOREM 1.1.1.</u> Let S be a subsemigroup of T_X, containing all constant maps. Then the automorphism group of S can be identified with the

subgroup of the automorphism group of $\mathcal{T}_X$, which leaves the subsemigroup S invariant. $\square$

R.P. Sullivan [23] substantially generalized this result to semigroups of partial transformations and binary relations.

For a transformation f of X we denote the range of f by R(f) (= f(X)). The *rank* of f is:

$$\text{rank } f = |R(f)|,$$

the cardinality of the range of f. An application of Theorem 1.1.1 is given by A.I. Mal'cev [14]. He characterised all ideals of $\mathcal{T}_X$ by showing that each has the form:

$$I_\xi = \{f \in \mathcal{T}_X \mid \text{rank } f < \xi\},$$

where ξ is a cardinal not exceeding the immediate successor $|X|^+$ of $|X|$, such that $\xi > 1$. Clearly,

$$\mathcal{C}_X \subseteq I_\xi, \quad \text{for every } \xi.$$

Hence each I_ξ has only inner automorphisms.

From semigroups containing constant maps we move to another wide class of semigroups, $\mathcal{G}_X$*-normal* semigroups. These are semigroups invariant under conjugation by all bijections of X. There is considerable variation within the class of $\mathcal{G}_X$-normal semigroups from subgroups of $\mathcal{G}_X$ to semigroups containing neither constants nor 1-1 or onto transformations. We refer to $\mathcal{G}_X$-normal subgroups of $\mathcal{G}_X$ as *normal* subgroups. The study of automorphisms of $\mathcal{G}_X$-normal semigroups attracted the attention of a number of mathematicians, being an interesting problem on its own, and also providing a starting point for a study of automorphisms of arbitrary semigroups of transformations.

If X is a finite set and S is a $\mathcal{G}_X$-normal subsemigroup of $\mathcal{G}_X$, then S is a normal subgroup of $\mathcal{G}_X$ [22, 1.6.4]. Such subgroups were described by Galois [6]. Recall that any bijection, or permutation, f of X can be written as a product of either an even or odd number of 2-cycles. Accordingly f is called either *even* or *odd*. The set of all even SEE ERRATA (in pocket at back) permutations of X is a normal subgroup of G_X [22, 10.4.5] called the *alternating group* on X and is denoted by Alt_X. If S is a normal subgroup of G_X then $S = \{\iota_X\}$ or $S = \mathrm{Alt}_X$ or $S = G_X$, where ι_X is the identity transformation of X, except when $|X| = 4$, when S can also be the *Klein 4-group* K_4,

$$K_4 = \{\iota_X, (12)(34), (13)(24), (14)(23)\}$$

(see [22, 10.8.8] for $|X| > 4$). A discussion of automorphisms of normal subgroups of G_X can be found in [22, 11.4.6 and 11.4.8], and we present a summary of this in the following theorem.

<u>THEOREM 1.1.2</u>. If X is a finite set, $|X| \neq 6$, then all normal subgroups of G_X have only inner automorphisms. If $|X| = 6$, then G_X and Alt_X have outer automorphisms. □

All G_X-normal subsemigroups of T_X, the full transformation semigroup on X, where X is a finite set, were described by J.S.V. Symons [25]. In particular he showed that any G_X-normal subsemigroup of T_X either contains C_X, the set of all constant maps of X, or is a normal subgroup of G_X. In view of Theorems 1.1.1 and 1.1.2 this implies

<u>THEOREM 1.1.3</u>. Let X be a finite set and S be a G_X-normal semigroup. If $|X| \neq 6$, or $|X| = 6$ and $S \neq \mathrm{Alt}_X$ or G_X, then all automorphisms of S are inner. If $|X| = 6$ and $S = \mathrm{Alt}_X$ or G_X, then S has outer automorphisms.

□

That result completes our discussion of automorphisms of G_X-normal semigroups of transformations of a finite set X.

Now let X be an infinite set.

<u>REMARK 1.1.4</u>. Any G_X-normal subsemigroup S of G_X is in fact a subgroup of G_X. Indeed, if $f \in S$, then f and f^{-1} have the same number of n-cycles

(for $1 < n \in \mathbb{N}$ or $n = \infty$), hence $f^{-1} = hfh^{-1}$, for some $h \in \mathcal{G}_X$ [22, 1.3.11], so that $f^{-1} \in S$. □

Normal subgroups of $\mathcal{G}_X$ were described by Schreier and Ulam [20] for denumerably infinite X and by Baer [1] for arbitrary infinite X.

For a transformation f of X the set $S(f) = \{x \in X | f(x) \neq x\}$ is called the *shift* of f. Take an infinite cardinal α not exceeding $|X|^+$. Then

$$\mathrm{Sym}(X,\alpha) = \{f \in \mathcal{G}_X \mid |S(f)| < \alpha\}$$

is a normal subgroup of $\mathcal{G}_X$ [22, 11.2.1].

A permutation of X with a finite shift is called a *finite* permutation. If f is a finite permutation, then $f|_{S(f)}$, the restriction of f to the shift of f, is either even or odd. The set of finite even permutations on X forms a normal subgroup of $\mathcal{G}_X$, which is called the *alternating group* Alt_X [22, 11.2.2].

If S is a normal subgroup of $\mathcal{G}_X$, then $S = \{\iota_X\}$ or $S = \mathrm{Alt}_X$ or $S = \mathrm{Sym}(X,\alpha)$, $\alpha \leqslant |X|$, or $S = \mathcal{G}_X$ (see, for example, [22, 11.3.4]). The following description of automorphisms of normal subgroups of $\mathcal{G}_X$ can be found in [22, 11.4.6].

<u>THEOREM 1.1.5.</u> Let X be an infinite set and S be a $\mathcal{G}_X$-normal subgroup of $\mathcal{G}_X$. Then all automorphisms of S are inner. □

Now let S be a $\mathcal{G}_X$-normal subsemigroup of $\mathcal{T}_X$ (X is an infinite set). So far we have given descriptions of automorphisms of S when either $C_X \subseteq S$ or $S \subseteq \mathcal{G}_X$. In 1975 Fitzpatrick and Symons [5] presented the following result which considerably extended knowledge about automorphisms of $\mathcal{G}_X$-normal semigroups.

<u>THEOREM 1.1.6.</u> Let X be an infinite set and S be a $\mathcal{G}_X$-normal semigroup of transformations of X. If S contains $\mathcal{G}_X$, then all automorphisms of S are inner. □

We present an outline of the proof of the theorem above. Every automorphism of S induces an inner (Theorem 1.1.5) automorphism of G_X. This and a simple trick due to Schreier [21] allows to reduce the proof to showing that if an automorphism ϕ of S is the identity on G_X then ϕ is also the identity on $S - G_X$. The proof of the latter based on two lemmas.

1. If f in S is 1-1, $x,y,u,v \in X$ with $x \neq y$ and $u \neq v$, then

$$(u,v)f(x,y) = f \quad \text{iff} \quad f(\{x,y\}) = \{u,v\}.$$

(Here (x,y) denotes the permutation of X interchanging x and y.) This readily implies that

$$f(\{x,y\}) = \phi(f)(\{x,y\}),$$

and hence $f = \phi(f)$.

A *partition* $\pi(f)$ of a transformation f of X is a set $\{f^{-1}(x) \mid x \in R(f)\}$.

2. If f in S is not 1-1, A and B in $\pi(f)$ and $x,y \in X$, then

$$\{f(A),f(B)\} = \{x,y\} \quad \text{iff} \quad \forall g \in S,\ g(x) = g(y) \text{ implies } gf(A) = gf(B).$$

Now the fact that $f = \phi(f)$ can be easily deduced with the aid of the observation that

$$f(x) = f(y) \quad \text{iff} \quad f(x,y) = f \quad \text{iff} \quad \phi(f)(x) = \phi(f)(y).$$

An extension of Theorem 1.1.6 is due to Sullivan [24] who showed that any G_X-normal semigroup of 1-1 transformations (total or partial) containing the 3-cycles (and hence the alternating group Alt_X) possesses only inner automorphisms.

The first step in the study of automorphisms of constant-free G_X-normal semigroups which do not necessarily contain bijections, was made by Schein [18], [19]. He studied G_X-normal semigroups of 1-1 transformations and presented the following result.

THEOREM 1.1.7. Let X be an infinite set and S be a G_X-normal semigroup of 1-1 transformations of X. Then all automorphisms of S are inner. □

A special case of Theorem 1.1.7 for Baer-Levi semigroups can be found in [8].

Schein's proof of Theorem 1.1.7, we understand, is based on the same ideas as the proof of the result in [8]. To outline the latter we use the following notions. A bijection H of a subset $\mathcal{A}$ of the power set $\mathcal{P}_X$ of X is called an *order-automorphism* of $\mathcal{A}$, if H and H^{-1} both preserve the natural ordering of $\mathcal{A}$ given by set inclusion. An order-automorphism H of $\mathcal{A}$ is said to be *induced* by an h in G_X, if

$$H(A) = h(A), \text{ for all } A \in \mathcal{A}.$$

The key points of the proof of the result in [8] are:

1. Every $\phi \in \text{Aut } S$ determines an order-automorphism H of the set R(S) of ranges of all transformation in S via

$$H(R(f)) = R(\phi(f)).$$

We say that H is the *order-automorphism of* ϕ.

2. Every order-automorphism of R(S) is induced.

Thus, in particular, the order-automorphism H of ϕ is induced by some $h \in G_X$.

3. For every $x \in X$ there exist A and B in R(S) with $A \subseteq B$ and $B - A = \{x\}$.

Hence if $f \in S$, then

$$f(x) = f(B-A) = R(fv) - R(fu),$$

where $u, v \in S$ with $R(u) = A$ and $R(v) = B$. With the aid of some simple manipulations and the fact that the action of H is determined by h it is shown then that $\phi(f)h(x) = hf(x)$, and hence

$$\phi(f) = hfh^{-1}, \text{ for all } f \in S.$$

Thus every $\phi \in$ Aut S is inner.

Now, a subset $\mathcal{A}$ of $\mathcal{P}_X$ is said to be *normal* if

$$h(\mathcal{A}) = \mathcal{A}, \text{ for all } h \in \mathcal{G}_X,$$

where $h(\mathcal{A}) = \{h(A) \mid A \in \mathcal{A}\}$. Certainly, the set of ranges R(S) of an arbitrary $\mathcal{G}_X$-normal semigroup S is normal. Attempting to generalize the above proof to an arbitrary $\mathcal{G}_X$-normal semigroup S, we started by studying order-automorphisms of R(S). This resulted in a characterisation of all normal subsets of $\mathcal{P}_X$ which have only induced order-automorphisms (Chapter 5 and [12]). Subsequently, we found an extensive class of $\mathcal{G}_X$-normal semigroups, which have non-induced order-automorphisms of their families of ranges.

EXAMPLE 1.1.8. Fix $n \in \mathbb{N}$, $n > 1$ and let

$$S = \{f \in \mathcal{T}_X \mid |X - R(f)| = n \text{ and } |f^{-1}(x)| = |X| \text{ for every } x \in R(f)\}.$$

Certainly, S is a $\mathcal{G}_X$-normal semigroup. Since

$$R(fg) = R(f), \text{ for all } f,g \in S,$$

R(S) consists of all $A \in \mathcal{P}_X$ with $|A'| = n$, where $A' = X - A$. Theorem 5.3.13 ensures there exist non-induced order-automorphisms of R(S). Moreover, for every A and B in R(S), $A \subseteq B$ implies $A = B$. This illustrates the fact that a technique via statement 3 in the above outline of the proof of the result in [8] fails to be of any use for this semigroup. □

These observations encourage us to invent a new technique for studying automorphisms of $\mathcal{G}_X$-normal semigroups (Chapter 2 and [11]). The essence is the production of certain maximal right and left ideals. With this we proved the following:

THEOREM 1.1.9. Every G_X-normal semigroup has only inner automorphisms. □

This result subsumes all previously stated results for automorphisms of G_X-normal semigroups (on an infinite set X).

A complete description of automorphisms of Croisot-Teissier semigroups, which are generally not G_X-normal, and contain no constants and no bijections is given in Chapter 3. More detailed discussion on the technique used and the results obtained is given in Section 3.0.

1.2. RANGE FAMILIES OF TRANSFORMATION SEMIGROUPS

The problem of describing precisely those subsets of P_X which serve as sets of ranges of transformation semigroups is another subject of our study (Problem 2, Section 1.0). We mentioned in Section 1.0 that the only results on the matter have been obtained by P.M. Olonichev [17]. Here we present his characterization of the families of ranges of monogenic semigroups of partial transformations.

By a *monogenic* semigroup S, generated by a transformation f (total or partial) we mean a semigroup consisting of all the powers of f, and denote it by

$$S = \langle f \rangle (= \{f, f^2, f^3, \ldots\}) .$$

If all the powers of f are different, that is

$$f^m = f^n \text{ implies } m = n ,$$

then we say that S is of *infinite type*. Otherwise, there exists the smallest natural number m such that

$$f^m = f^n (n > m) . \tag{1.2.1}$$

Then S is said to have *type* (m,n), where n is the smallest natural number satisfying (1.2.1). Let Φ denote the empty set. The next two theorems are due to P.M. Olonichev [17].

THEOREM 1.2.1. A set $\{A_i \subseteq X \mid i = 1, \ldots, n\}$ is the set of ranges of a monogenic semigroup of type (m,n) of partial transformations if and only if either (i) each $A_i \neq \Phi$ and

a) $A_1 \supsetneq A_2 \supsetneq \ldots \supsetneq A_m = A_{m+1} = \ldots = A_n$;

b) $|A_m| \geqslant n - m$;

c) $|A_1'| \geqslant |A_1 - A_2| \geqslant \ldots \geqslant |A_{m-1} - A_m|$;

or (ii) $A_k = \Phi$ some $k \leqslant n$ and $A_i \neq \Phi (i < k)$, and

a) $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_k$;

b) $m = k = n - 1$;

$$\text{c)} \quad |A_1'| \geqslant |A_1 - A_2| \geqslant \ldots \geqslant |A_{m-1} - A_m| \,. \qquad \square$$

THEOREM 1.2.2. A set $\{A_i \subseteq X \mid i = 1,2,\ldots\}$ is the set of ranges of a monogenic semigroup of the infinite type of partial transformations if and only if the following four conditions are satisfied:

(i) $|A_i| \geqslant \aleph_0$, each i ;

(ii) $A_i \supseteq A_{i+1}$, each i ;

(iii) $A_i = A_{i+1}$ implies $A_i = A_{i+k}$, each $k \in \mathbb{N}$;

(iv) $|A_1'| \geqslant |A_1 - A_2| \geqslant |A_2 - A_3| \geqslant \ldots$. $\square$

In Chapter 4 (see also [13]) we characterise those normal subsets of $\mathcal{P}_X$ which serve as sets of ranges of semigroups of total transformations of X, and constant-free semigroups of total transformations of X. We also show that each of the above subsets serve as the set of ranges of some $\mathcal{G}_X$-normal semigroup of total transformations.

CHAPTER 2

AUTOMORPHISMS OF NORMAL TRANSFORMATION SEMIGROUPS

2.0. INTRODUCTION AND PRELIMINARIES

A transformation semigroup S over X is said to be G_X-*normal* if

$$hSh^{-1} = S, \text{ for all } h \in G_X .$$

The full transformation semigroup T_X, the semigroups of all 1-1 and all onto transformations and the group G_X itself, are examples of G_X-normal semigroups.

If S is a G_X-normal semigroup, then for each $h \in G_X$, the map ϕ of S given by

$$\phi(f) = hfh^{-1} \quad (f \in S)$$

is an automorphism of S, specifically an *inner automorphism* of S. Our purpose is to prove the following

THEOREM 2.0.1. Every automorphism of a G_X-normal semigroup is inner. □

The subject of this Chapter was suggested to the author by G.R. Wood.

To prove the above theorem we produce certain maximal right (Section 2.1) and left (Section 2.2) ideals. We note a remarkable duality between properties of these right and left ideals.

For the purpose of our proof we partition all G_X-normal semigroups into three types:

1. Semigroups containing a constant map;

and constant-free semigroups into:

2. Semigroups of 1-1 transformations;

and

3. Constant-free semigroups containing a transformation which is not 1-1.

All automorphisms of semigroups of the first type are inner [23, Theorem 1], so we can restrict our attention to constant-free semigroups.

We begin with some general notes on G_X-normal semigroups.

For a function $f : X \to X$ we denote the *range* of f by $R(f)$ $(= f(X))$ and the *partition* of f by $\pi(f)$ $(= \{f^{-1}(x) \mid x \in R(f)\})$.

If S is an arbitrary semigroup of transformations, let

$$R(S) = \{R(f) \mid f \in S\} \quad \text{and} \quad \pi(S) = \{\pi(f) \mid f \in S\}.$$

We say that $R(S)$ $(\pi(S))$ is *normal* if for each $h \in G_X$

$$h(R(S)) = R(S) \quad (h(\pi(S)) = \pi(S)),$$

(by $h(R(S))$ we mean $\{h(A) \mid A \in R(S)\}$ and by $h(\pi(S))$ we mean $\{h(\mathcal{A}) \mid \mathcal{A} \in \pi(S)\}$ where $h(\mathcal{A}) = \{h(A) \mid A \in \mathcal{A}\}$).

LEMMA 2.0.2. If S is a G_X-normal semigroup, then $R(S)$ and $\pi(S)$ are normal. The proof is straightforward. □

We say that a semigroup is *trivial* if $S = \{\iota_X\}$, where ι_X is the identity transformation of X. In what follows S is non-trivial.

PROPOSITION 2.0.3. Every G_X-normal semigroup S is transitive.

Proof. Take arbitrary x,y in X. We construct f in S such that $f(x) = y$.

Firstly let x and y be distinct and suppose there exists a $g \in S$ with $g(x) = z \neq x$. If $z = y$ we let $f = g$, otherwise $(y,z)g(y,z)$ is the required f. To construct g, observe that since S is non-trivial there exists a $q \in S$ together with distinct u and v in X such that $q(u) = v$. If $u = x$ we let $g = q$, otherwise $g = (u,x)q(u,x)$.

Now suppose $y = x$, choose any p in S and let $p(x) = w$. If $w = x$ we let $f = p$. Otherwise choose $t \in S$ with $t(w) = x$ (using the first part of the proof), then $f = tp$ takes x to x as required. □

<u>REMARK 2.0.4.</u> We exclude from our consideration G_X-normal subsemigroups of G_X, since they are all subgroups of G_X, and hence have only inner automorphisms (Remark 1.1.4 and Theorem 1.1.5). □

2.1. G_X-NORMAL SEMIGROUPS OF 1-1 TRANSFORMATIONS

In this section S denotes a G_X-normal semigroup of 1-1 transformations

DEFINITION 2.1.1. Let $x \in X$ and

$$\mathcal{R}_x = \{r \in S \mid x \in X - R(r)\} .$$

Then $\mathcal{R}_x$ is a right ideal of S, which we call a *point right ideal.* □

We will use the following observation based on the normality of R(S) (Lemma 2.0.2) and the fact that S is not a subsemigroup of G_X, that is R(S) contains proper subsets of X.

REMARK 2.1.2. Given $x,y \in X$ with $x \neq y$ there exists an A in R(S) with $x \in X - A$ and $y \in A$. □

LEMMA 2.1.3. Given $x,y \in X$ the following three statements are equivalent:

(i) $\mathcal{R}_x \subseteq \mathcal{R}_y$;

(ii) $x = y$;

(iii) $\mathcal{R}_x = \mathcal{R}_y$.

Proof. Implications (ii) ⇒ (iii) and (iii) ⇒ (i) are trivial. We show (i) ⇒ (ii). Suppose $x \neq y$ and choose an $A \in R(S)$ with $x \in X - A$, $Y \in A$ (Remark 2.1.2). If $f \in S$ with $R(f) = A$, then $f \in \mathcal{R}_x - \mathcal{R}_y$, so $\mathcal{R}_x \nsubseteq \mathcal{R}_y$, proving (i) ⇒ (ii). □

Define a map $\theta : X \to \{\mathcal{R}_x \mid x \in X\}$ via $\theta(x) = \mathcal{R}_x$, each $x \in X$.

LEMMA 2.1.4. θ is a bijection.

Proof. Clearly θ is onto and Lemma 2.1.3 ensures θ is 1-1. □

DEFINITION 2.1.5. Given distinct $f_1, f_2 \in S$ let

$$\mathcal{R}_{f_1,f_2} = \{r \in S \mid f_1 r = f_2 r\} .$$

Then $\mathcal{R}_{f_1,f_2}$ is a right ideal of S (possibly empty), which we call a *function right ideal.* □

We will show (Result 2.1.8) that there always exist distinct f_1, f_2 in S such that $\mathcal{R}_{f_1,f_2}$ is non-empty. However $\mathcal{R}_{f_1,f_2}$ may be empty. Observe that given f_1 and f_2,

$$r \in \mathcal{R}_{f_1,f_2} \quad \text{iff} \quad R(r) \subseteq \{x \in X \mid f_1(x) = f_2(x)\}.$$

Hence if we choose f_1 and f_2 which are never equal, then $\mathcal{R}_{f_1,f_2} = \Phi$. Let S, for example, be the Baer-Levi semigroup of type $(|X|,|X|)$ [8], that is the semigroup of all 1-1 transformations f such that $|R(f)| = |X - R(f)| = |X|$. Note that S is $\mathcal{G}_X$-normal and choose $f_1 \in S$, then $X - R(f_1) \in R(S)$ (Lemma 2.0.2). If $f_2 \in S$ with $R(f_2) = X - R(f_1)$, then $\mathcal{R}_{f_1,f_2} = \Phi$.

The following notation applies to an arbitrary $\mathcal{G}_X$-normal semigroup S.

<u>NOTATION 2.1.6.</u> Let f_1, f_2 be distinct transformations in S. Then

$$\mathcal{D}_{f_1,f_2} = \{x \in X \mid f_1(x) \neq f_2(x)\}$$

and

$$D_{f_1,f_2} = \{\{f_1(x), f_2(x)\} \mid x \in \mathcal{D}_{f_1,f_2}\}. \qquad \square$$

Returning to semigroups of 1-1 transformations, we now derive relationships between point right ideals and function right ideals.

<u>PROPOSITION 2.1.7.</u> Let $f_1, f_2 \in S$ with $\mathcal{R}_{f_1,f_2} \neq \Phi$. Then

$$\mathcal{R}_{f_1,f_2} = \bigcap_{x \in \mathcal{D}_{f_1,f_2}} \mathcal{R}_x .$$

<u>Proof.</u> Let $r \in \mathcal{R}_{f_1,f_2}$, that is $f_1 r = f_2 r$. If $x \in \mathcal{D}_{f_1,f_2}$, or $f_1(x) \neq f_2(x)$, then $x \in X - R(r)$, so $r \in \mathcal{R}_x$, and since this is true for each $x \in \mathcal{D}_{f_1,f_2}$ we conclude that $r \in \bigcap_{x \in \mathcal{D}_{f_1,f_2}} \mathcal{R}_x$ or

$$\mathcal{R}_{f_1,f_2} \subseteq \bigcap_{x \in \mathcal{D}_{f_1,f_2}} \mathcal{R}_x .$$

Conversely, if $r \in \bigcap_{x \in \mathcal{D}_{f_1,f_2}} \mathcal{R}_x$, then for each y in R(r) we have

$y \in X - \mathcal{D}_{f_1,f_2}$, or $f_1(y) = f_2(y)$ and hence $f_1 r = f_2 r$, that is $r \in \mathcal{R}_{f_1,f_2}$, so

$$\bigcap_{x \in \mathcal{D}_{f_1,f_2}} \mathcal{R}_x \subseteq \mathcal{R}_{f_1,f_2} ,$$

which proves the desired equality. □

PROPOSITION 2.1.8. Given $x \in X$ there exist $f_1, f_2 \in S$ such that $\mathcal{R}_x = \mathcal{R}_{f_1,f_2}$.

Proof. On account of Proposition 2.1.7 it is sufficient to construct f_1, f_2 such that $\mathcal{D}_{f_1,f_2} = \{x\}$.

Observe that there exists an f in S with

$$|X - R(f)| \geqslant 2 .$$

(For an arbitrary f in $S - G_X$,

$$|X - R(f^2)| = |X - R(f)| + |X - R(f)| ,$$

and we replace f with f^2).

Using the normality of $R(S)$ (Lemma 2.0.2), choose an f in S with

$$x \in X - R(f) \quad \text{and} \quad |X - R(f)| \geqslant 2.$$

Let $f(x) = y$ and $z \in X - R(f)$, $z \neq x$. If

$$g = (x,z)f(x,z) ,$$

then $g(z) = y$ and $z \in X - R(g)$. We let

$$h = (y,z), \; f_1 = gf \quad \text{and} \quad f_2 = hgh^{-1}f .$$

Then for each $u \neq x$:

$$\begin{aligned} f_1(u) = gf(u) &= gh^{-1}f(u), && \text{since } f(u) \neq y \text{ for } u \neq x \\ &&& \text{and } z \notin R(f) ; \\ &= hgh^{-1}f(u), && \text{since } gh^{-1}f(u) \neq y \\ &&& \text{for } f(u) \neq y \\ &&& \text{and } z \notin R(g) ; \\ &= f_2(u). \end{aligned}$$

However

$$f_1(x) = gf(x) = g(y)$$

while

$$f_2(x) = hgh^{-1}f(x) = hgh^{-1}(y) = hg(z) = h(y) = z \neq g(y) ,$$

since $z \in X - R(g)$. Hence $f_1(x) \neq f_2(x)$ and $\mathcal{D}_{f_1,f_2} = \{x\}$. $\square$

PROPOSITION 2.1.9. Given f_1 and f_2 in S, $\mathcal{R}_{f_1,f_2}$ is a maximal function right ideal if and only if $|\mathcal{D}_{f_1,f_2}| = 1$.

Proof. Suppose $\mathcal{R}_{f_1,f_2}$ is a maximal function right ideal, while $x,y \in \mathcal{D}_{f_1,f_2}$, $x \neq y$. Then

$$\begin{aligned} \mathcal{R}_{f_1,f_2} &= \bigcap_{z \in \mathcal{D}_{f_1,f_2}} \mathcal{R}_z && \text{(Proposition 2.1.7)} \\ &\subseteq \mathcal{R}_x \cap \mathcal{R}_y \\ &\subsetneq \mathcal{R}_x && \text{(Lemma 2.1.3)}. \end{aligned}$$

It follows from Proposition 2.1.8 that there exist g_1 and g_2 with

$$\mathcal{R}_{g_1,g_2} = \mathcal{R}_x ,$$

and so

$$\mathcal{R}_{f_1,f_2} \subsetneq \mathcal{R}_x = \mathcal{R}_{g_1,g_2} ,$$

a contradiction to the maximality of $\mathcal{R}_{f_1,f_2}$. Hence $|\mathcal{D}_{f_1,f_2}| = 1$.

For the converse, suppose $\mathcal{D}_{f_1,f_2} = \{x\}$, some $x \in X$, while there exist $g_1,g_2 \in S$ such that

$$\mathcal{R}_{g_1,g_2} \supseteq \mathcal{R}_{f_1,f_2} .$$

Since $\mathcal{R}_{g_1,g_2} = \bigcap_{y \in \mathcal{D}_{g_1,g_2}} \mathcal{R}_y$ (Proposition 2.1.7) we have

$$\bigcap_{y \in \mathcal{D}_{g_1,g_2}} \mathcal{R}_y = \mathcal{R}_{g_1,g_2} \supseteq \mathcal{R}_{f_1,f_2} = \mathcal{R}_x \quad \text{(Proposition 2.1.7 again)},$$

and so Lemma 2.1.3 ensures $\mathcal{D}_{g_1,g_2} = \{x\}$, that is

$$\mathcal{R}_{g_1,g_2} = \mathcal{R}_x = \mathcal{R}_{f_1,f_2} . \qquad \square$$

COROLLARY 2.1.10. Given f_1 and f_2 in S,

$\mathcal{R}_{f_1,f_2}$ is a maximal function right ideal if and only if $\mathcal{R}_{f_1,f_2} = \mathcal{R}_x$, some $x \in X$.

Proof. Follows from Propositions 2.1.7 and 2.1.9. $\square$

We show now that each automorphism ϕ of S permutes point right ideals.

PROPOSITION 2.1.11. Given $x \in X$,

$$\phi(\mathcal{R}_x) = \mathcal{R}_y ,$$

for some $y \in X$.

Proof. Choose f_1 and f_2 in S such that $\mathcal{R}_{f_1,f_2} = \mathcal{R}_x$ (Proposition 2.1.8), then

$$\begin{aligned}
\phi(\mathcal{R}_x) = \phi(\mathcal{R}_{f_1,f_2}) &= \phi(\{r \mid f_1 r = f_2 r\}) \\
&= \{\phi(r) \mid \phi(f_1 r) = \phi(f_2 r)\} \\
&= \{\phi(r) \mid \phi(f_1)\phi(r) = \phi(f_2)\phi(r)\} \\
&= \{r' \mid \phi(f_1)r' = \phi(f_2)r'\} \\
&= \mathcal{R}_{\phi(f_1),\phi(f_2)} .
\end{aligned}$$

Now Corollary 2.1.10 ensures $\mathcal{R}_{f_1,f_2}$ is a maximal function right ideal, hence $\mathcal{R}_{\phi(f_1),\phi(f_2)}$ $(= \phi(\mathcal{R}_{f_1,f_2}))$ is a maximal function right ideal, so there exists $y \in X$ such that

$$\mathcal{R}_{\phi(f_1),\phi(f_2)} = \mathcal{R}_y \qquad \text{(Corollary 2.1.10)}$$

and thus

$$\phi(\mathcal{R}_x) = \mathcal{R}_{\phi(f_1),\phi(f_2)} = \mathcal{R}_y . \qquad \square$$

Define a map

$$\eta : \{\mathcal{R}_x \mid x \in X\} \to \{\mathcal{R}_x \mid x \in X\}$$

via $\eta(\mathcal{R}_x) = \phi(\mathcal{R}_x)$, each $\mathcal{R}_x \subseteq S$.

LEMMA 2.1.12. η is a bijection.

Proof. That η is a mapping is the content of Proposition 2.1.11. Similarly by considering the automorphism ϕ^{-1} we define a map

$$\zeta : \{\mathcal{R}_x \mid x \in X\} \to \{\mathcal{R}_x \mid x \in X\}$$

via $\zeta(\mathcal{R}_x) = \phi^{-1}(\mathcal{R}_x)$, each $\mathcal{R}_x \subseteq S$.
Certainly, ζ is the inverse of η and so η is a bijection. □

We now define a map

$$h : X \to X \text{ via } h(x) = y, \text{ where } \eta(\mathcal{R}_x) = \mathcal{R}_y, \text{ each } x \in X .$$

It is clear, that

$$h = \theta^{-1}\eta\theta ,$$

and so Lemmas 2.1.4 and 2.1.12 ensure h is a bijection of X. We call h the *bijection associated with* ϕ .

LEMMA 2.1.13. Given $f \in S$,

$$R(\phi(f)) = h(R(f)) .$$

Proof. Observe that to show $R(\phi(f)) = h(R(f))$ it is sufficient to show that

$$X - R(\phi(f)) = h(X - R(f)) ,$$

because for the bijection h, $h(X - R(f)) = X - h(R(f))$.

Now if $x \in X - R(f)$, that is $f \in \mathcal{R}_x$, then $\phi(f) \in \eta(\mathcal{R}_x) = \mathcal{R}_{h(x)}$, so $h(x) \in X - R(\phi(f))$, or

$$h(X - R(f)) \subseteq X - R(\phi(f)).$$

To show the reverse inclusion is true, observe that $h^{-1} = \theta^{-1}\eta^{-1}\theta$ is the bijection associated with ϕ^{-1} and so the first part of the proof implies that given $g \in S$,

$$h^{-1}(X - R(g)) \subseteq X - R(\phi^{-1}(g)) .$$

In particular taking $g = \phi(f)$ we have $h^{-1}(X - R(\phi(f))) \subseteq X - R(\phi^{-1}(\phi(f)))$, or

$$h(X - R(f)) \supseteq X - R(\phi(f)) ,$$

and the equality follows. □

We complete our study of automorphisms of G_X-normal semigroups of 1-1 transformations, that is, semigroups of Type 2, by presenting the following result.

PROPOSITION 2.1.14. Let S be a G_X-normal semigroup of 1-1 transformations ($S \nsubseteq G_X$). Then each automorphism ϕ of S is inner, that is, for some $h \in G_X$

$$\phi(f) = hfh^{-1}, \text{ for each } f \in S.$$

Proof. Consider the bijection h associated with ϕ as defined prior to Lemma 2.1.13. Take an arbitrary $f \in S$, $x \in X$ and let $f(x) = y$. Choose A in R(S) with $A \neq X$ and $x \in A$. Let $z \in X - A$ and $B = (A - \{x\}) \cup \{z\} \in R(S)$ (Lemma 2.0.2). Choose p and q in S such that $R(p) = A$ and $R(q) = B$.

Now $R(p) - R(q) = A - B = \{x\}$, thus $R(fp) - R(fq) = \{f(x)\} = \{y\}$. Using Lemma 2.1.13 we have:

$$R(\phi(p)) - R(\phi(q)) = \{h(x)\}$$

and

$$R(\phi(fp)) - R(\phi(fq)) = \{h(y)\} .$$

However

$$\begin{aligned} R(\phi(fp)) - R(\phi(fq)) &= R(\phi(f)\phi(p)) - R(\phi(f)\phi(q)) \\ &= \{\phi(f)h(x)\} , \end{aligned}$$

so

$$\phi(f)h(x) = h(y) = hf(x), \text{ that is}$$

$$\phi(f) = hfh^{-1} .$$ □

<u>REMARK 2.1.15</u>. The fact that every G_X-normal semigroup of 1-1 transformations possesses only inner automorphisms was first established by B.M. Schein ([18], [19] and Theorem 1.1.7). We understand that his proof, based on the study of ordered sets of ranges, is quite different from ours. □

2.2. G_X-NORMAL CONSTANT-FREE SEMIGROUPS CONTAINING A TRANSFORMATION WHICH IS NOT 1-1

Let S be a G_X-normal constant-free semigroup containing a transformation which is not 1-1. We prove that all automorphisms of S are inner. We start by showing that R(S) contains only sets of cardinality $|X|$.

LEMMA 2.2.1. If S is a G_X-normal constant-free semigroup, then $|R(f)| = |X|$, each $f \in S$.

Proof. Suppose there is an f in S with $|R(f)| = \alpha < |X|$, that is $|\pi(f)| = |R(f)| = \alpha$. We show that there exists an $A \in \pi(f)$ with $|A| \geqslant \alpha$. The result is clear when α is finite. Hence assume α is infinite and denote by α^+ the cardinal successor of α. Then either $\alpha^+ = |X|$ (and so $|X|$ is regular [16,21.14]) or there exists $\beta < |X|$, $\beta = \alpha^+$ (and so β is regular [16,21.14]). The assumption that each $A \in \pi(f)$ has a cardinality less than α implies that $|\cup \pi(f)| < |X|$ or $|\cup \pi(f)| < \beta < |X|$ respectively [16,21.18], a contradiction. Hence we can choose an $A \in \pi(f)$ with $|A| \geqslant \alpha$ and a $B \in R(S)$ with $B \subseteq A$ and $|B| = \alpha$ (Lemma 2.0.2) together with a $g \in S$ such that $R(g) = B$. Then $|R(fg)| = 1$, so that fg is a constant map in S, a contradiction which proves $|R(f)| = |X|$. □

Let P_2 be the set of all doubletons in X.

DEFINITION 2.2.2. Given $A \in P_2$, $A = \{a_1, a_2\}$, let

$$\mathcal{L}_A = \{\ell \in S \mid \ell(a_1) = \ell(a_2)\}.$$

Then $\mathcal{L}_A$ is a left ideal of S which we call a *point left ideal*. □

LEMMA 2.2.3. For each $A \in P_2$, $\mathcal{L}_A \neq \Phi$.

Proof. Choose a map f in S which is not 1-1, say $f(x) = f(y)$ for distinct $x, y \in X$. If $h \in G_X$ is such that $\{h(x), h(y)\} = A$ then $hfh^{-1} \in \mathcal{L}_A$. □

LEMMA 2.2.4. Given $A,B \in P_2$, the following three statements are equivalent:

(i) $\mathcal{L}_A \subseteq \mathcal{L}_B$;

(ii) $A = B$;

(iii) $\mathcal{L}_A = \mathcal{L}_B$.

Proof. Implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are trivial. We show (i) $\Rightarrow$ (ii). Let $B = \{b_1,b_2\}$ and suppose $A \neq B$, say $b_1 \in B - A$. Choose an $\ell \in \mathcal{L}_A$ (Lemma 2.2.3) and let $x \in R(\ell) - \ell(A \cup B)$ (note: $|X| = |R(\ell)| > |\ell(A \cup B)|$, Lemma 2.2.1). If $y \in X$ is such that $\ell(y) = x$, let $h = (b_1,y)$ and $f = h\ell h^{-1}$. We show $f \in \mathcal{L}_A - \mathcal{L}_B$. That $f \in \mathcal{L}_A$ follows from the fact that h moves only points b_1 and y, which are not in A.

To show that $f \notin \mathcal{L}_B$, observe that

$f(b_1) = h\ell h^{-1}(b_1) = h\ell(y) = h(x)$, while

$f(b_2) = h\ell h^{-1}(b_2) = h\ell(b_2)$, because $b_2 \neq y$ (else $x = \ell(y) = \ell(b_2) = \ell(B)$, contrary to the choice of x) ;

$\neq h(x)$, because $\ell(b_2) = \ell(B) \neq x$.

Thus $f(b_1) \neq f(b_2)$ and $f \notin \mathcal{L}_B$. □

Define a map $\delta : P_2 \to \{\mathcal{L}_A \mid A \in P_2\}$ via $\delta(A) = \mathcal{L}_A$, each $A \in P_2$.

LEMMA 2.2.5. δ is a bijection.

Proof. Clearly δ is onto and Lemma 2.2.4 ensures δ is 1-1. □

DEFINITION 2.2.6. Given distinct $f_1,f_2 \in S$ let

$$\mathcal{L}_{f_1,f_2} = \{\ell \in S \mid \ell f_1 = \ell f_2\} .$$

Then $\mathcal{L}_{f_1,f_2}$ is a left ideal of S (possibly empty, see Example 2.2.7 below), which we call a *function left ideal.* □

We will show (Proposition 2.2.10) that for each G_X-normal constant-free semigroup S containing a transformation which is not 1-1 there exist $f_1,f_2 \in S$ with $\mathcal{L}_{f_1,f_2} \neq \Phi$. In general, the question of whether $f_1,f_2 \in S$

generate a non-empty $\mathcal{L}_{f_1,f_2}$ is the question of whether the equation $\ell f_1 = \ell f_2$ has a solution ℓ in S. The example below illustrates that $\mathcal{L}_{f_1,f_2}$ may be empty.

EXAMPLE 2.2.7. Let S be the dual Baer-Levi semigroup of the type $(|X|,|X|)$ [2], that is the semigroup of all onto mappings f such that $|f^{-1}(x)| = |X|$ for each $x \in X$. Certainly S is $\mathcal{G}_X$-normal. Assume $X = \mathbb{N}$, so that $|X| = \aleph_0$. Fix an arbitrary $f_1 \in S$ and let $\mathcal{A} = \pi(f_1) = \{A_1, A_2, A_3, \ldots\}$. Partition each $A_i \in \mathcal{A}$ such that $A_i = A_i' \dot{\cup} A_i''$, $|A_i'| = |A_i''| = \aleph_0$. Let $\mathcal{B}$ be the partition of X given by $\mathcal{B} = \{A_1', A_1'' \dot{\cup} A_2', A_2'' \dot{\cup} A_3', \ldots\}$. Since $\mathcal{B}$ is a partition of X into $\aleph_0$ sets, each of cardinality $\aleph_0$, $\mathcal{B} \in \pi(S)$, and so there exists $f_2 \in S$ with $\pi(f_2) = \mathcal{B}$. Suppose $\ell \in \mathcal{L}_{f_1,f_2}$ that is $\ell f_1 = \ell f_2$ and let $\ell f_1(A_1) = x$. Then because of the choice of $\mathcal{B}$ we have the following chain of equalities:

$x = \ell f_1(A_1) = \ell f_1(A_1'') = \ell f_2(A_1'') = \ell f_2(A_2') = \ell f_1(A_2') = \ell f_1(A_2) = \ldots$

thus $x = \ell f_1(A_1) = \ell f_1(A_2) = \ldots$, that is $R(\ell f_1) = \{x\}$ and ℓf_1 is a constant in S, contradicting the construction of S, so that $\mathcal{L}_{f_1,f_2} = \Phi$. □

Recall that $\mathcal{D}_{f_1,f_2}$ and D_{f_1,f_2} (Notation 2.1.6) were defined for an arbitrary $\mathcal{G}_X$-normal semigroup S $(f_1,f_2 \in S)$. The following remark is an immediate consequence of the definition of D_{f_1,f_2}.

REMARK 2.2.8. Let $f_1,f_2 \in S$, then $D_{f_1,f_2} \subseteq \mathcal{P}_2$. □

We proceed with two results deriving relationships between point left ideals and function left ideals.

PROPOSITION 2.2.9. Let f_1 and f_2 be distinct elements of S, and $\mathcal{L}_{f_1,f_2} \neq \Phi$. Then

$$\mathcal{L}_{f_1,f_2} = \bigcap_{A \in D_{f_1,f_2}} \mathcal{L}_A .$$

Proof. Let $\ell \in \mathcal{L}_{f_1,f_2}$, that is, $\ell f_1 = \ell f_2$ and so for each $x \in \mathcal{D}_{f_1,f_2}$ we have $\ell f_1(x) = \ell f_2(x)$ (recall $f_1(x) \neq f_2(x)$) so $\ell \in \mathcal{L}_{\{f_1(x),f_2(x)\}}$ and since this is true for each $x \in \mathcal{D}_{f_1,f_2}$ we conclude

$$\ell \in \bigcap_{x \in \mathcal{D}_{f_1,f_2}} \mathcal{L}_{\{f_1(x),f_2(x)\}} = \bigcap_{A \in D_{f_1,f_2}} \mathcal{L}_A \text{ , or } \mathcal{L}_{f_1,f_2} \subseteq \bigcap_{A \in D_{f_1,f_2}} \mathcal{L}_A .$$

Conversely, let $\ell \in \bigcap_{A \in D_{f_1,f_2}} \mathcal{L}_A$, then for each $x \in \mathcal{D}_{f_1,f_2}$, $\ell f_1(x) = \ell f_2(x)$. Now for each $y \notin \mathcal{D}_{f_1,f_2}$ we have $f_1(y) = f_2(y)$, so we deduce $\ell f_1 = \ell f_2$. That is,

$$\ell \in \mathcal{L}_{f_1,f_2} \text{ and } \bigcap_{A \in D_{f_1,f_2}} \mathcal{L}_A \subseteq \mathcal{L}_{f_1,f_2} ,$$

which proves the desired equality. □

PROPOSITION 2.2.10. Given an $A \in \mathcal{P}_2$, there exist f_1 and f_2 in S such that

$$\mathcal{L}_A = \mathcal{L}_{f_1,f_2} .$$

Proof. On account of Proposition 2.2.9 it is sufficient to construct f_1 and f_2 such that

$$D_{f_1,f_2} = \{A\} .$$

Choose an f in $\mathcal{L}_A$ (Lemma 2.2.3) and let $f(A) = z$. Let $A = \{a_1,a_2\}$. Since S is transitive (Proposition 2.0.3) there exists g in S such that $g(z) = a_1$. Let $h = (a_1,a_2)$ and

$$f_1 = gf \quad ; \quad f_2 = hf_1h^{-1} .$$

Since h moves only points in A and $f_1 \in \mathcal{L}_A$ ($\mathcal{L}_A$ is a left ideal), we conclude that $f_2 = hf_1$. For each $x \in X - f_1^{-1}(A)$ we have:

$$f_1(x) = hf_1(x) = f_2(x) ,$$

so $\mathcal{D}_{f_1,f_2} \subseteq f_1^{-1}(A)$. Now if $x \in f_1^{-1}(A)$, that is $f_1(x) = a_i$, $i = 1,2$, then

$$f_1(x) = a_i \neq h(a_i) = hf_1(x) = f_2(x) ,$$

hence $\mathcal{D}_{f_1,f_2} \supseteq f_1^{-1}(A)$. We conclude

$$\mathcal{D}_{f_1,f_2} = f_1^{-1}(A) .$$

Thus

$$\begin{aligned} D_{f_1,f_2} &= \{\{f_1(x),f_2(x)\} \mid x \in \mathcal{D}_{f_1,f_2}\} && \text{(Notation 2.1.6)} \\ &= \{\{f_1(x),f_2(x)\} \mid x \in f_1^{-1}(A)\} \\ &= \{\{a_i,h(a_i)\} \mid i = 1,2\} \\ &= \{\{a_1,a_2\}\} \\ &= \{A\} , \end{aligned}$$

as required. □

PROPOSITION 2.2.11. Given distinct f_1 and f_2 in S, $\mathcal{L}_{f_1,f_2}$ is a maximal function left ideal if and only if $|D_{f_1,f_2}| = 1$.

Proof. Let $\mathcal{L}_{f_1,f_2}$ be a maximal function left ideal and suppose $A,B \in D_{f_1,f_2}$, $A \neq B$. Then $A,B \in \mathcal{P}_2$ (Remark 2.2.8). Hence

$$\begin{aligned} \mathcal{L}_{f_1,f_2} &= \bigcap_{C \in D_{f_1,f_2}} \mathcal{L}_C && \text{(Proposition 2.2.9)} \\ &\subseteq \mathcal{L}_A \cap \mathcal{L}_B \\ &\subsetneq \mathcal{L}_A && \text{(Lemma 2.2.4)} \\ &= \mathcal{L}_{g_1,g_2} && \text{(Proposition 2.2.10),} \end{aligned}$$

for some distinct $g_1,g_2 \in S$, contradicting the maximality of $\mathcal{L}_{f_1,f_2}$. Hence $|D_{f_1,f_2}| = 1$.

Conversely, suppose $D_{f_1,f_2} = \{A\}$, some $A \in \mathcal{P}_2$, while there exists a function left ideal $\mathcal{L}_{g_1,g_2}$ $(g_1,g_2 \in S)$ such that

$$\mathcal{L}_{g_1,g_2} \supseteq \mathcal{L}_{f_1,f_2} .$$

Since $\mathcal{L}_{g_1,g_2} = \bigcap_{B \in D_{g_1,g_2}} \mathcal{L}_B$ (Proposition 2.2.9) we have

$$\bigcap_{B \in D_{g_1,g_2}} \mathcal{L}_B = \mathcal{L}_{g_1,g_2} \supseteq \mathcal{L}_{f_1,f_2} = \mathcal{L}_A \quad \text{(Proposition 2.2.9 again)},$$

and so Lemma 2.2.4 ensures $D_{g_1,g_2} = \{A\}$, that is

$$\mathcal{L}_{g_1,g_2} = \mathcal{L}_A = \mathcal{L}_{f_1,f_2} . \qquad \square$$

<u>COROLLARY 2.2.12</u>. Given f_1 and f_2 in S,

$\mathcal{L}_{f_1,f_2}$ is a maximal left function ideal if and only if $\mathcal{L}_{f_1,f_2} = \mathcal{L}_A$, some $A \in \mathcal{P}_2$.

<u>Proof</u>. Follows from Propositions 2.2.9 and 2.2.11. $\square$

We show now that each automorphism ϕ of S permutes point left ideals.

<u>PROPOSITION 2.2.13</u>. Given $A \in \mathcal{P}_2$,

$$\phi(\mathcal{L}_A) = \mathcal{L}_B ,$$

for some $B \in \mathcal{P}_2$.

<u>Proof</u>. Choose f_1 and f_2 in S such that $\mathcal{L}_{f_1,f_2} = \mathcal{L}_A$ (Proposition 2.2.10), then

$$\begin{aligned}
\phi(\mathcal{L}_A) = \phi(\mathcal{L}_{f_1,f_2}) &= \phi(\{\ell \mid \ell f_1 = \ell f_2\}) \\
&= \{\phi(\ell) \mid \phi(\ell f_1) = \phi(\ell f_2)\} \\
&= \{\phi(\ell) \mid \phi(\ell)\phi(f_1) = \phi(\ell)\phi(f_2)\} \\
&= \{\ell' \mid \ell'\phi(f_1) = \ell'\phi(f_2)\} \\
&= \mathcal{L}_{\phi(f_1),\phi(f_2)} .
\end{aligned}$$

Now Corollary 2.2.12 ensures $\mathcal{L}_{f_1,f_2}$ is a maximal function left ideal, hence $\mathcal{L}_{\phi(f_1),\phi(f_2)}$ $(= \phi(\mathcal{L}_{f_1,f_2}))$ is a maximal function left ideal, so there exists $B \in \mathcal{P}_2$ such that

$$\mathcal{L}_{\phi(f_1),\phi(f_2)} = \mathcal{L}_B \quad \text{(Corollary 2.2.12)}.$$

We conclude

$$\phi(\mathcal{L}_A) = \mathcal{L}_{\phi(f_1),\phi(f_2)} = \mathcal{L}_B . \qquad \square$$

Define a map

$$\mu : \{\mathcal{L}_A \mid A \in P_2\} \to \{\mathcal{L}_A \mid A \in P_2\}$$

via $\mu(\mathcal{L}_A) = \phi(\mathcal{L}_A)$, each $\mathcal{L}_A \subseteq S$.

<u>LEMMA 2.2.14.</u> μ is a bijection.

<u>Proof.</u> That μ is a mapping is the content of Proposition 2.2.13. Similarly by considering the automorphism ϕ^{-1} we define a map

$$\xi : \{\mathcal{L}_A \mid A \in P_2\} \to \{\mathcal{L}_A \mid A \in P_2\}$$

via $\xi(\mathcal{L}_A) = \phi^{-1}(\mathcal{L}_A)$, each $\mathcal{L}_A \subseteq S$.
Certainly, ξ is the inverse of μ and so μ is a bijection. □

We now define a map
$\lambda : P_2 \to P_2$ via $\lambda(A) = B$, where $\mu(\mathcal{L}_A) = \mathcal{L}_B$, each $A \in P_2$.

It is clear that

$$\lambda = \delta^{-1}\mu\,\delta ,$$

and so Lemmas 2.2.5 and 2.2.14 ensure λ is a bijection of P_2. We call λ the *bijection of* P_2 *associated with* ϕ.

We show that λ is *induced* by a bijection h of X, that is

$$\lambda(A) = h(A) ,$$

for each $A \in P_2$. Note here that not every bijection of P_2 is induced, as shown in Example 2.2.15 below.

<u>EXAMPLE 2.2.15.</u> Fix A and C in P_2, $A \neq C$ and let λ be a bijection of P_2, which interchanges A and C and the identity otherwise. Choose $B \in P_2$, $B = \{x,y\}$ such that $x \in A - C$ and $y \in X - (A \cup C)$. Note $A \cap B = \{x\}$ and $B \cap C = \Phi$. Suppose λ is induced by $h \in G_X$, then
$h(x) = h(A \cap B) = h(A) \cap h(B) = \lambda(A) \cap \lambda(B) = C \cap B = \Phi$.
Thus λ is not induced. □

Observe that in the example above we had λ, a bijection of P_2, such that

$$|A \cap B| \neq |\lambda(A) \cap \lambda(B)| ,$$

for some A,B in P_2. This leads us to a criterion for a bijection λ of P_2 to be induced.

PROPOSITION 2.2.16. Let λ be a bijection of P_2. Then λ is induced if and only if $|A \cap B| = |\lambda(A) \cap \lambda(B)|$, for every $A,B \in P_2$.

Proof. If λ is induced by an $h \in G_X$, then for every $A,B \in P_2$ $|A \cap B| = |h(A \cap B)| = |h(A) \cap h(B)| = |\lambda(A) \cap \lambda(B)|$.

For the converse suppose that λ is a bijection of P_2 such that for every $A,B \in P_2$

$$(2.2.1) \qquad |A \cap B| = |\lambda(A) \cap \lambda(B)| .$$

We show that λ is induced. This is done in the following three steps.

1. *Given* $x \in X$ *there exists a unique* $y \in X$ *such that for every* $A,B \in P_2$ *with* $A \cap B = \{x\}$ *we have* $\lambda(A) \cap \lambda(B) = \{y\}$.

Take a pair A,B in P_2 with $A \cap B = \{x\}$, then by (2.2.1) $\lambda(A) \cap \lambda(B) = \{y\}$, for some $y \in X$.

Take any other pair C,D in P_2 with $|C \cap D| = 1$ and let $F \subseteq P_2$ be such that:

α) for every distinct $F_1,F_2 \in F$, $|F_1 \cap F_2| = 1$;

β) for any $F \in F$, $|A \cap F| = |B \cap F| = |C \cap F| = |D \cap F| = 1$.

We show:

$C \cap D = \{x\}$ iff there exists an F (as described above) with $|F| = |X|$.

Let $A \cup B \cup C \cup D = E$, then $|E| \leqslant 8$ and $|X - E| = |X|$.

Assume firstly that $C \cap D = \{x\}$ and let $F = \{\{x,y\} \mid y \in X - E\}$. Then F satisfies α) and β) and $|F| = |X - E| = |X|$.

For the converse assume $C \cap D = \{z\}$, $z \neq x$ and $F \subseteq P_2$ satisfies α) and β). For each $F \in F$ we have $|E \cap F| > 1$. (If not, then

$$|E \cap F| = |(A \cup B \cup C \cup D) \cap F|$$
$$= |(A \cap F) \cup (B \cap F) \cup (C \cap F) \cup (D \cap F)| \leqslant 1 .$$

Using condition β) we conclude:

$$A \cap F = B \cap F = C \cap F = D \cap F = A \cap B = \{x\} ,$$

or $C \cap D = \{x\}$, a contradiction).

Define a map $\nu : \mathcal{F} \to \mathcal{P}_E$, where $\mathcal{P}_E$ is the power set of E, via $\nu(F) = E \cap F$, each $F \in \mathcal{F}$. We show ν is 1-1. Suppose $F_1, F_2 \in \mathcal{F}$ with $\nu(F_1) = \nu(F_2)$. Then

$$1 < |E \cap F_1| = |E \cap F_1 \cap F_2| \leqslant |F_1 \cap F_2| ,$$

so that $|F_1 \cap F_2| > 1$, thus $F_1 = F_2$ (condition α)). However $\mathcal{P}_E$ is finite, so $|\mathcal{F}| \leqslant |\mathcal{P}_E| < \aleph_0$, or $|\mathcal{F}| < |X|$. We conclude $C \cap D = \{x\}$.

Observe now that the definition of the set $\mathcal{F}$ depends on the sets A,B,C and D. We denote this dependence by $\mathcal{F} = \mathcal{F}(A,B,C,D)$. Hence

$C \cap D = \{x\}$ iff $\exists \mathcal{F}(A,B,C,D)$ with $|\mathcal{F}(A,B,C,D)| = |X|$

iff $\exists \mathcal{F}(\lambda(A),\lambda(B),\lambda(C),\lambda(D))$ with $|\mathcal{F}(\lambda(A),\lambda(B),\lambda(C),\lambda(D))| = |$

(assumption (2.2.1))

iff $\lambda(C) \cap \lambda(D) = \{y\}$.

Now define a map

$h : X \to X$ via $\{h(x)\} = \lambda(A) \cap \lambda(B)$, where $\{x\} = A \cap B$, for $A,B \in \mathcal{P}_2$ and each $x \in X$.

2. h *is a bijection of* X.

That h is well-defined is the content of Step 1. Observe that the bijection λ^{-1} of $\mathcal{P}_2$ is associated with the automorphism ϕ^{-1}. By considering ϕ^{-1} and λ^{-1} instead of ϕ and λ we define a map

$k : X \to X$ via $\{k(x)\} = \lambda^{-1}(A) \cap \lambda^{-1}(B)$, where $\{x\} = A \cap B$, for $A,B \in \mathcal{P}_2$ and each $x \in X$.

Then for each $x \in X$

$$\begin{aligned}\{kh(x)\} &= k(\lambda(A) \cap \lambda(B)), \quad \text{where} \quad A \cap B = \{x\} \\ &= \lambda^{-1}\lambda(A) \cap \lambda^{-1}\lambda(B) \\ &= A \cap B \\ &= \{x\} .\end{aligned}$$

Similarly we can show $hk(x) = x$, for each $x \in X$. Thus k is the inverse of h, and so h is a bijection of X.

3. *λ is induced by* h.

To show λ is induced by h we must show $\lambda(A) = h(A)$ for each $A \in \mathcal{P}_2$. From the definition of h we at once have $h(A) \subseteq \lambda(A)$. Take $y \in \lambda(A)$ and let $B \in \mathcal{P}_2$ be such that $\lambda(A) \cap \lambda(B) = \{y\}$. Then $A \cap B = \{x\}$, some $x \in A$, so $h(x) = y$ and $h(A) \supseteq \lambda(A)$. The equality follows. □

<u>REMARK 2.2.17</u>. In view of Proposition 2.2.16 our aim now is to show that for every $A,B \in \mathcal{P}_2$ (2.2.1) holds for λ associated with ϕ as defined prior to Example 2.2.15. Observe that (2.2.1) is equivalent to the statement

(2.2.2) $|A \cap B| = 1$ if and only if $|\lambda(A) \cap \lambda(B)| = 1$, for each $A,B \in \mathcal{P}_2$.

Indeed (2.2.1) certainly implies (2.2.2). We show the reverse implication.

Assume (2.2.2) holds. If $|A \cap B| = 2$, that is $A = B$, then $\lambda(A) = \lambda(B)$, and so $|\lambda(A) \cap \lambda(B)| = 2$. If $|A \cap B| = 1$, then by our assumption $|\lambda(A) \cap \lambda(B)| = 1$. The case $|A \cap B| = 0$ follows by elimination. □

<u>REMARK</u>. The proof of Proposition 2.2.16 is presented for the sake of completeness. This result is actually a special case of Theorem 5.1.12 since (2.2.1) and (2.2.2) are equivalent (Remark 2.2.17). □

The next lemma illustrates the fact that the existence of a transformation f in S which is not 1-1 provides an extensive variety of elements in $\pi(S)$.

LEMMA 2.2.18. Given $B_1, B_2 \subseteq X$ with $B_1 \cap B_2 = \Phi$ and $|B_1| = |B_2| = 3$ there exists an $\mathcal{A} \in \pi(S)$ with $B_1 \subseteq A_1 \in \mathcal{A}$, $B_2 \subseteq A_2 \in \mathcal{A}$.

Proof. Suppose that there exists a transformation f in S such that:

$$C_1, C_2 \in \pi(f) \quad \text{and} \quad |C_1|,\ |C_2| \geqslant 3 .$$

Choose a bijection p of X with

$$B_1 \subseteq p(C_1) \quad \text{and} \quad B_2 \subseteq p(C_2) .$$

Certainly $pfp^{-1} \in S$. Let

$$\mathcal{A} = \pi(pfp^{-1}) \ (= p(\pi(f))),\ A_1 = p(C_1) \quad \text{and} \quad A_2 = p(C_2) ,$$

then $A_1, A_2 \in \mathcal{A} \in \pi(S)$ and $B_1 \subseteq A_1$, $B_2 \subseteq A_2$.

To construct such an f as used above we first show that there exists a g in S such that

$$g(x_1) = g(x_2) = g(x_3) = x_1, \text{ for some distinct } x_1, x_2, x_3 \in X.$$

Choose a t in S not 1-1 and let $x, x_1, x_2 \in X$ be such that

$$t(x_1) = t(x_2) = x .$$

We assume $x = x_1$ (for if $x \neq x_1$ choose $s \in S$ such that $s(x) = x_1$ (Proposition 2.0.3) and replace t by st). Let $x_4 \in R(t) - \{t^{-1}(x_1)\}$ (note: $R(t) - \{t^{-1}(x_1)\} \neq \Phi$, else t^2 is a constant in S) and let $x_3 \in X$ such that $t(x_3) = x_4$. Then

$$g = (x_2, x_4)t(x_2, x_4)t$$

is such that $g(x_1) = g(x_2) = g(x_3) = x_1$.

To accomplish the construction of the above f choose distinct z_1, z_2, z_3 in $R(g) - \{g^{-1}(x_1)\}$ together with $y_1, y_2, y_3 \in X$ such that $g(y_i) = z_i$, $i = 1,2,3$. Let

$$k = (x_1, z_1)(x_2, z_2)(x_3, z_3) \in \mathcal{G}_X \quad \text{and} \quad f = kgk^{-1}g .$$

Let $kg(z_1) = z_4$. Then

$$f(x_1) = f(x_2) = f(x_3) = z_4$$

and

$$f(y_1) = f(y_2) = f(y_3) = z_1 .$$

Now $z_1 \neq z_4$ (else $kg(z_1) = z_1$ implies $g(z_1) = x_1$ or $z_1 \in g^{-1}(x_1)$, contrary to the choice of z_1). Let $C_1 = f^{-1}(z_1)$, $C_2 = f^{-1}(z_4)$. Then $|C_1|, |C_2| \geqslant 3$ and $C_1, C_2 \in \pi(f)$ as required. $\square$

REMARK 2.2.19. It easily follows from Lemma 2.2.18 that

$$\mathcal{L}_A \cap \mathcal{L}_B \neq \Phi ,$$

for every $A,B \in P_2$. $\square$

LEMMA 2.2.20. Let $A,B \in P_2$, $A \neq B$. Then $|A \cap B| = 1$ iff there is a C in P_2, $C \neq A$ or B, such that $\mathcal{L}_A \cap \mathcal{L}_B \subseteq \mathcal{L}_C$.

Proof. Assume $|A \cap B| = 1$ and let $C = (A \cup B) - (A \cap B)$.
For each ℓ in $\mathcal{L}_A \cap \mathcal{L}_B$ (Remark 2.2.19):

$$\ell(A) = \ell(A \cap B) = \ell(B) = \ell(A \cup B) = \ell(C) ,$$

so that $\ell \in \mathcal{L}_C$ and $\mathcal{L}_A \cap \mathcal{L}_B \subseteq \mathcal{L}_C$.

For the converse suppose $A \cap B = \Phi$ and $C \in P_2$ is distinct from A and B. Let $C = \{c_1, c_2\}$. Since $|A \cap C| \leqslant 1$ and $|B \cap C| \leqslant 1$ assume without loss of generality that $c_1 \in X - B$ and $c_2 \in X - A$. Choose $A \in \pi(S)$ with $A \cup \{c_1\} \subseteq A_1 \in A$, $B \cup \{c_2\} \subseteq A_2 \in A$ and $A_1 \neq A_2$ (Lemma 2.2.18).

If $\ell \in S$ has $\pi(\ell) = A$, then $\ell \in (\mathcal{L}_A \cap \mathcal{L}_B) - \mathcal{L}_C$.
This confirms that $|A \cap B| = 1$. $\square$

LEMMA 2.2.21. Let A,B and C be distinct elements of P_2. Then

$$\mathcal{L}_A \cap \mathcal{L}_B \subseteq \mathcal{L}_C \text{ iff } \mathcal{L}_{\lambda(A)} \cap \mathcal{L}_{\lambda(B)} \subseteq \mathcal{L}_{\lambda(C)} .$$

Proof. Observe that $\mathcal{L}_A \cap \mathcal{L}_B \neq \Phi$ (Remark 2.2.19) and $\mathcal{L}_A \cap \mathcal{L}_B \subseteq \mathcal{L}_C$ iff $\phi(\mathcal{L}_A \cap \mathcal{L}_B) \subseteq \phi(\mathcal{L}_C)$.

Now

$$\phi(\mathcal{L}_A \cap \mathcal{L}_B) = \phi(\mathcal{L}_A) \cap \phi(\mathcal{L}_B) = \mathcal{L}_{\lambda(A)} \cap \mathcal{L}_{\lambda(B)},$$

by the definition of λ. Also $\phi(\mathcal{L}_C) = \mathcal{L}_{\lambda(C)}$, so that

$$\phi(\mathcal{L}_A \cap \mathcal{L}_B) \subseteq \phi(\mathcal{L}_C) \quad \text{iff} \quad \mathcal{L}_{\lambda(A)} \cap \mathcal{L}_{\lambda(B)} \subseteq \mathcal{L}_{\lambda(C)},$$

and the desired equivalence is established. □

PROPOSITION 2.2.22. Given A and B in P_2,

$$|A \cap B| = 1 \text{ if and only if } |\lambda(A) \cap \lambda(B)| = 1.$$

Proof. We have:

$|A \cap B| = 1$ iff $\exists C \neq A$ or B such that $\mathcal{L}_A \cap \mathcal{L}_B \subseteq \mathcal{L}_C$ (Lemma 2.2.20)

iff $\exists\lambda(C) \neq \lambda(A)$ or $\lambda(B)$ such that $\mathcal{L}_{\lambda(A)} \cap \mathcal{L}_{\lambda(B)} \subseteq \mathcal{L}_{\lambda(C)}$ (λ is a bijection and Lemma 2.2.21)

iff $|\lambda(A) \cap \lambda(B)| = 1$ (Lemma 2.2.20 again). □

From Propositions 2.2.16, 2.2.22 and Remark 2.2.17 we readily deduce

PROPOSITION 2.2.23. λ is induced by a bijection of X. □

Now we are ready to show that a constant-free G_X-normal semigroup containing a transformation which is not 1-1 (that is a semigroup of Type 3) possesses only inner automorphisms.

PROPOSITION 2.2.24. Let S be a constant-free G_X-normal semigroup containing a transformation which is not 1-1. Then each automorphism ϕ of S is inner, that is for some $h \in G_X$

$$\phi(f) = hfh^{-1}, \text{ for each } f \in S.$$

Proof. Let h be the bijection which induces λ (Proposition 2.2.23). In what follows we use the fact that for any distinct $x_1, x_2 \in X$

$$\phi(\mathcal{L}_{\{x_1,x_2\}}) = \mathcal{L}_{\lambda(\{x_1,x_2\})} = \mathcal{L}_{\{h(x_1),h(x_2)\}}.$$

Take an arbitrary $f \in S$, $x \in X$ and let $y \in X$ with $f(x) \neq f(y)$ (that is $f \notin \mathcal{L}_{\{x,y\}}$). Then

$$\phi(\mathcal{L}_{\{f(x),f(y)\}}) = \mathcal{L}_{\{hf(x),hf(y)\}} .$$

Let $\phi(g) \in \phi(\mathcal{L}_{\{f(x),f(y)\}})$. Then $g \in \mathcal{L}_{\{f(x),f(y)\}}$ or $gf(x) = gf(y)$. It follows that $\phi(gf) \in \mathcal{L}_{\{h(x),h(y)\}}$, hence

$$\phi(g)\phi(f)h(x) = \phi(g)\phi(f)h(y) .$$

Note that $f \notin \mathcal{L}_{\{x,y\}}$ implies $\phi(f) \notin \phi(\mathcal{L}_{\{x,y\}})$ or $\phi(f) \notin \mathcal{L}_{\{h(x),h(y)\}}$ that is

$$\phi(f)h(x) \neq \phi(f)h(y) .$$

Thus $\phi(g) \in \mathcal{L}_{\{\phi(f)h(x),\phi(f)h(y)\}}$ and we conclude

$$\phi(\mathcal{L}_{\{f(x),f(y)\}}) \subseteq \mathcal{L}_{\{\phi(f)h(x),\phi(f)h(y)\}} .$$

This in turn implies

$$\mathcal{L}_{\{hf(x),hf(y)\}} \subseteq \mathcal{L}_{\{\phi(f)h(x),\phi(f)h(y)\}} .$$

Hence $\{hf(x),hf(y)\} = \{\phi(f)h(x),\phi(f)h(y)\}$ (Lemma 2.2.4).

Since the choice of y is independent of x (providing $y \neq x$) we conclude

$$\phi(f)h(x) = hf(x), \text{ for each } x \in X ,$$

so that

$$\phi(f) = hfh^{-1}, \text{ as required .}$$ □

CONCLUSION

We return to

THEOREM 2.0.1. Every automorphism of a $\mathcal{G}_X$-normal semigroup S is inner.

Proof. If S is a semigroup of Type 1, that is, contains a constant transformation, then we appeal to Sullivan [23, Theorem 1].

If S is a semigroup of Type 2, that is, a semigroup of 1-1 transformations, the result is given in 2.1.14 and 2.0.4.

If S is a semigroup of Type 3, that is, a semigroup containing a transformation which is not 1-1, then the result is given in 2.2.24.

This completes the proof of Theorem 2.0.1. □

REMARK. If X is a *finite* set and S is a semigroup of transformations of X which is not contained in G_X, then S is a G_X-normal if and only if all automorphisms of S are inner [25].

However, this is not the case for an infinite set X. While, as we showed, every G_X-normal semigroup S has only inner automorphisms, there are examples (Chapter 3 and [10]) of semigroups which are neither subsemigroups of G_X, nor G_X-normal, yet have only inner automorphisms. □

REMARK. Let A be a normal subset of P_X which can serve as a range family for some G_X-normal semigroup S (see Chapter 4 for the characterization of such A's). Let H be an order-automorphism of A. If H is the order-automorphism of some $\phi \in$ Aut S, then Theorem 2.0.1 implies that H is induced. Conversely, if H is induced by some $h \in G_X$, then a map $\phi : S \to S$ given by $f \mapsto hfh^{-1}$ is an automorphism of S, and H is the order-automorphism of ϕ. We conclude that an order-automorphism H of A is induced if and only if H is the order-automorphism of some $\phi \in$ Aut S. □

CHAPTER 3

AUTOMORPHISMS OF CROISOT-TEISSIER SEMIGROUPS

3.0. INTRODUCTION AND PRELIMINARIES

Croisot-Teissier semigroups, introduced in the early 1950's, have importance as natural containers of all simple, idempotent-free semigroups with a minimal left ideal. Our purpose in this Chapter is to offer a complete description of all automorphisms of such semigroups, a task suggested by B.M. Schein. A rich variety of automorphisms is found, ranging from inner, to "locally" inner, to thoroughly outer.

We adhere to the Clifford and Preston definition of a Croisot-Teissier semigroup, $S = CT(X,\mathcal{E},p,q)$, as detailed later in this section. It is an idempotent-free semigroup of transformations of the set X and is a union $S = \bigcup_{i\in I} T_i$ of left ideals associated with the family $\mathcal{E} = \{\mathcal{E}_i\}_{i\in I}$ of equivalences on X. When $p = q$, S is simple, so $S = S^2$. Of interest in our work is the fact that when $p > q$, S may not equal S^2, a property which allows a richer automorphism structure. A recent variation on the definition [15] produces certain simple subsemigroups of the general version. These could be handled by our techniques for the case $p = q$.

A by-product of our work is a canonical covering of S by means of right ideals, $S = \bigcup_{\alpha\in\Omega} I_\alpha$, independent of any automorphism, and determined by certain maximal families of ranges. This parallels the familiar left ideal decomposition. The manner in which automorphisms permute the components of both these natural decompositions of S is fundamental to our analysis.

Our results are presented in five sections: with each section a further ingredient is required for the description of the automorphism. Within each the main theorem is presented early and the proof produced via a series of lemmas and propositions. We proceed to an informal statement of our results. For basic Croisot-Teissier terminology, the reader is directed to the latter part of this section. Let ϕ be an automorphism of $S = CT(X,\mathcal{E},p,q)$ and W be the union of all well-separated sets.

In Section 3.1 we assume that $X = W$ (equivalently, S acts transitively on X) and that W is *reduced*, that is, $\bigcap_{i\in I} \mathcal{E}_i = \iota_W$, the identity relation on W. Then ϕ is inner. That is,

$$\phi(f) = hfh^{-1}, \text{ for all } f \in S,$$

where h is a fixed partition-preserving (p.p.) bijection of W (Theorem 3.1.2).

In Section 3.2 we assume only that $X = W$. Then ϕ is *locally inner*, that is,

$$\phi(f) = h_\alpha f h_\alpha^{-1}, \text{ for all } f \in I_\alpha,$$

where $S = \bigcup_{\alpha\in\Omega} I_\alpha$ is the right ideal decomposition mentioned previously, and $\{h_\alpha\}_{\alpha\in\Omega}$ is a suitably compatible system of p.p. bijections of W (Theorem 3.2.2). This is best possible, in the sense that all automorphisms are inner if and only if there is only one component in the right ideal decomposition (Theorem 3.2.35). A further consequence is that if $|\mathcal{E}| < p$, then all automorphisms are inner (Corollary 3.2.36). We remark that Baer-Levi semigroups are reduced, with $X = W$ and $|\mathcal{E}| = 1$, so they fall into Section 3.1 and this special situation within Section 3.2. We note at this point that Corollary 3.5.17 characterises general Croisot-Teissier semigroups for which all automorphisms are inner.

In Section 3.3 we assume that W *provides $\mathcal{E}$ cross-sections*, that is, all $\mathcal{E}_i$-classes meet W. Here ϕ is described in terms of an associated automorphism ψ of $CT(W,\mathcal{E}',p,q)$, where $\mathcal{E}'$ is the family of partitions of W induced by $\mathcal{E}$, and an associated bijection z of the set $\mathcal{E}$ of equivalences. Thus the description relies on Section 3.2 and the additional ingredient z (Theorem 3.3.3).

For Section 3.4 we assume ϕ is *range-preserving*, that is, for all f,g in S $R(f) \subseteq R(g)$ iff $R(\phi(f)) \subseteq R(\phi(g))$, a property which held in previous sections. Now a description of ϕ involves an ψ and z as in Section 3.3, together with an ingredient to describe the action of $\phi(f)$ on those classes of its partition which do not meet W. This requires arbitrary bijections from the set of such classes of $\mathcal{E}_i$ to those of $z(\mathcal{E}_i)$ (Theorem 3.4.4).

In Section 3.5 the general case is considered. Here ϕ is uniquely a product

$$\phi = \phi_1 \circ \phi_2$$

of a range-preserving automorphism ϕ_1 (as in Section 3.4) and a specialised η-stabilizing automorphism ϕ_2, for a natural congruence η on S (Theorem 3.5.1). The additional ingredient necessary to describe ϕ_2 (and hence ϕ) is an arbitrary family of bijections, one of each η-class of S outside S^2 (Corollary 3.5.12). Thus the most general ϕ requires four ingredients for its description.

A final consequence (Theorem 3.5.14) of the main theorem is a characterisation of the situation in which all automorphisms are range-preserving, namely if and only if W is reduced, or W provides $\mathcal{E}$ cross-sections, or $S = S^2$. We point out in the course of the Chapter that Croisot-Teissier semigroups for which W is reduced can be characterised by means of the requirement that a certain natural congruence be the identity, and likewise when W provides $\mathcal{E}$ cross-sections. This suggests that automorphisms of semigroups for which these congruences are the identity might be profitably studied.

Now we present the definition of Croisot-Teissier semigroups, as found in Clifford and Preston [4, pp86-93]. Let p and q be infinite cardinals with $p \geqslant q$, and let X be a set with $|X| \geqslant p$.

Partitions for Croisot-Teissier functions are provided by a set

$\mathcal{E} = \{\mathcal{E}_i\}_{i \in I}$ of distinct equivalence relations on X such that $|X/\mathcal{E}_i| = p$ for all $i \in I$. Members of $\mathcal{E}$ themselves will sometimes be regarded as partitions of X, rather than relations on X, the context making the interpretation clear. Ranges of Croisot-Teissier functions are C_q sets, defined as follows. A subset B of X is said to be *well-separated (w.s.) by* $\mathcal{E}$ if $|B| = p$ and $\mathcal{E}_i \cap (B \times B) = \iota_B$, the identity relation on B, for all $i \in I$ (so distinct elements of B are in distinct $\mathcal{E}_i$-classes). For a cardinal r, with $p \geqslant r \geqslant q$, C_r denotes the set of all well-separated subsets A of X whose complement in some containing well-separated set B has cardinality r. Formally,

$$C_r = \{A \mid \text{for some w.s. } B,\ A \subseteq B \text{ and } |B - A| = r\}\ .$$ SEE ERRATA

Note that for $x \in A \in C_r$, $A - \{x\} \in C_r$.

When X contains a well-separated set, the *Croisot-Teissier semigroup on* X, $\mathcal{E}$ *of type* (p,q) is

$$CT(X,\mathcal{E},p,q) = \{f : X \to X \mid \pi(f) \in \mathcal{E} \text{ and } R(f) \in C_q\}$$

with the operation of function composition. Unlike Clifford and Preston we compose functions as $fg(x) = f(g(x))$.

A Croisot-Teissier semigroup $S = CT(X,\mathcal{E},p,q)$ thus constructed is idempotent-free, with a left ideal decomposition

$$S = \bigcup_{i \in I} T_i$$

where

$$T_i = \{f : X \to X \mid \pi(f) = \mathcal{E}_i \text{ and } R(f) \in C_q\}.$$

When $p = q$, the T_i are minimal left ideals and S is consequently simple.

In situations where the parent Croisot-Teissier semigroup is clear, for $p \geqslant r \geqslant q$ we use the notation

$$CT(p,r) = CT(X,\mathcal{E},p,r) \subseteq CT(X,\mathcal{E},p,q)\ .$$

PROPOSITION 3.0.1. (1) $CT(p,r)$ is an ideal of $CT(X,\mathcal{E},p,q)$ and $CT(p,s) \subseteq CT(p,r)$ for $p \geqslant s \geqslant r \geqslant q$.

(2) $CT(p,p)$ is the (unique) minimal ideal of $CT(X,\mathcal{E},p,q)$.

Proof. (1) is straightforward, while for (2) use Lemma 3.3.7 (1) or see [4, Theorem 8.11]. $\square$

REMARK 3.0.2. For completeness we now briefly present the variation on the definition of Croisot-Teissier semigroups due to Mielke [15]. Let p,q,X and $\mathcal{E}$ be as above. A w.s. B is said to be q-*well-separated* by $\mathcal{E}$ if the collection of all $\mathcal{E}_i$-classes of X which do not intersect B, has cardinal less than or equal to q, for each $i \in I$. Let

$$C^*_q = \{A \mid \text{for some } q\text{-well-separated } B,\ A \subseteq B \text{ with } |B - A| = q\}.$$

When X contains a q-well-separated set, let

$$CT^*(X,\mathcal{E},p,q) = \{f : X \to X \mid \pi(f) \in \mathcal{E} \text{ and } R(f) \in C^*_q\}.$$

The above defined semigroups, as found in [15] are simple, idempotent free and each $CT^*(X,\mathcal{E},p,q)$ is the union of its minimal right ideals

$$T^*_i = \{f : X \to X \mid \pi(f) = \mathcal{E}_i \text{ and } R(f) \in C^*_q\}.$$

It is easily checked that $CT^*(X,\mathcal{E},p,q) \subsetneq CT(X,\mathcal{E},p,q)$ for $p \neq q$, and that $CT^*(X,\mathcal{E},p,p) = CT(X,\mathcal{E},p,p)$. $\square$

For a general equivalence relation σ on a set Y, and $A \subseteq Y$, we let

$$A^c = \{y \in Y \mid y\ \sigma\ a \text{ for some } a \in A\}$$

and call it the σ-*closure* of A. If $A = A^c$, we say A is σ-*closed*. A *cross-section* of σ is any subset of Y comprising precisely one element of each σ-class. We use implicitly the simple fact that a function $f : X \to X$ is completely determined by its partition $\pi(f)$ and values on any cross-section of $\pi(f)$.

3.1. AUTOMORPHISMS WHEN W = X: REDUCED CASE

For a Croisot-Teissier semigroup $S = CT(X,\mathcal{E},p,q)$, we denote by W the union of all well-separated subsets of X. Alternatively

$$W = \bigcup_{f \in S} R(f).$$

In this section and the next we assume $W = X$, equivalently, S acts transitively on X. This is always the case, for example, when $|\mathcal{E}| < p$ (see Corollary 3.2.36). For the main result of this section, Theorem 3.1.2, we also assume that X is reduced in the following sense:

DEFINITION 3.1.1. With $\mathcal{E} = \{\mathcal{E}_i\}_{i \in I}$ the given family of equivalences on X, set

$$\rho = \bigcap_{i \in I} \mathcal{E}_i .$$

If $\rho = \iota_X$ (the identity relation on X) we say that X is *reduced* (relative to $\mathcal{E}$) or that S is *reduced*. □

When W = X and X is reduced, automorphisms of S are especially simple:

THEOREM 3.1.2. Let $S = CT(X,\mathcal{E},p,q)$ be a Croisot-Teissier semigroup such that W = X and X is reduced. Then each automorphism ϕ of S is inner, that is, for some fixed partition-preserving bijection h of X,

$$\phi(f) = hfh^{-1}, \text{ for all } f \in S. \qquad \square$$

The proof of the Theorem will be given after we have assembled the necessary machinery.

Recall that for infinite cardinals $p \geqslant r$ and a set X of cardinality p, the *Baer-Levi semigroup* BL(p,r) on X of type (p,r) consists of all 1-1 maps f from X into itself such that $|X - R(f)| = r$. As a straightforward example of a transitive, reduced S we have:

EXAMPLE 3.1.3. Let $p \geqslant q$ be infinite cardinals and let X be a set with $|X| = p$. Let $\mathcal{E} = \{\iota_X\}$. Then

$$CT(X,\mathcal{E},p,q) = \bigcup_{p \geqslant r \geqslant q} BL(p,r).$$

Clearly X is reduced. Also, here a well-separated set is any subset of X of cardinality p, so certainly W = X.

Notice that taking p = q, we get that BL(p,p) is a transitive, reduced Croisot-Teissier semigroup. Therefore Theorem 3.1.2 generalizes [8] in this case. □

A more typical example of a transitive, reduced Croisot-Teissier semigroup is the following in which there are no 1-1 functions (that is $\mathcal{E}_i \neq \iota_X \ \forall i$):

<u>EXAMPLE 3.1.4</u>. Let $X = C \cup D$ where $|X| = |C - D| = |D - C| = p$. For each pair c,d with $c \in C - D$ and $d \in D - C$, let $\mathcal{E}_{c,d}$ be the equivalence on X determined by the partition

$$\{\{c,d\}, \text{ singletons}\}.$$

Let $\mathcal{E} = \{\mathcal{E}_{c,d} \mid c \in C - D,\ d \in D - C\}$ and let $S = CT(X,\mathcal{E},p,q)$. Since now well-separated subsets of X are those of cardinality p contained in C or D, we have W = X. Clearly X is reduced. □

For a general Croisot-Teissier semigroup $S = CT(X,\mathcal{E},p,q)$, the bijections h of X which induce inner automorphisms of S are the ones which are partition-preserving in the following sense:

<u>DEFINITION 3.1.5</u>. Let $\mathcal{E}$ be a family of partitions (or equivalences) of a set X. A bijection $h : X \to X$ is termed *partition-preserving* (p.p.) relative to $\mathcal{E}$ if for all partitions $A \in \mathcal{E}$, both the partitions $\{h(C) \mid C \in A\}$ and $\{h^{-1}(C) \mid C \in A\}$ are also in $\mathcal{E}$. □

<u>PROPOSITION 3.1.6</u>. Let $S = CT(X,\mathcal{E},p,q)$. Let $h \in G_X$. Then the map

$$\phi_h : S \to S, \quad f \mapsto hfh^{-1} \quad \forall f \in S$$

is an automorphism of S if and only if h is partition-preserving relative to $\mathcal{E}$. Furthermore, if h and k are partition-preserving with $\phi_h = \phi_k$, then $h|_W = k|_W$.

<u>Proof</u>. Assume h is p.p. Then h maps well-separated sets to well-separated sets, and so $R(\phi_h(f)) = h(R(f)) \in \mathcal{C}_q$. Also $\pi(\phi_h(f)) = \{h(C) \mid C \in \pi(f)\} \in \mathcal{E}$ because h is p.p. Thus $\phi_h(f) \in S$ for all $f \in S$ and hence ϕ_h is a homomorphism from S into S. Similarly $\phi_{h^{-1}}$ maps S into S and is an inverse of ϕ_h. Thus $\phi_h \in \text{Aut } S$.

Conversely assume $\phi_h \in \text{Aut } S$. Let $A \in \mathcal{E}$. Choose $f,g \in S$ with $\pi(f) = A$ and $\pi(\phi_h(g)) = A$. Then $\{h(C) \mid C \in A\} = \pi(\phi_h(f)) \in \mathcal{E}$ and $\{h^{-1}(C) \mid C \in A\} = \{h^{-1}(h(D)) \mid D \in \pi(g)\} = \pi(g) \in \mathcal{E}$. Thus h is p.p.

For the final statement, let $x \in W$ and write

$$\{x\} = R(f) - R(g) \text{ for some } f,g \in S.$$

Then

$$\begin{aligned}\{h(x)\} &= h(R(f) - R(g)) \\ &= R(\phi(f)) - R(\phi(g)) \\ &= k(R(f) - R(g)) \\ &= \{k(x)\},\end{aligned}$$

so $h(x) = k(x)$. $\square$

<u>REMARK</u>. It is not sufficient to require only that $\{h(C) \mid C \in A\} \in \mathcal{E}$, for each $A \in \mathcal{E}$, in order for h to be p.p. For example consider $X = \mathbf{Z} \dot{\cup} \{b\}$, $I = \mathbf{N}$, $\mathcal{E}_i = \{\{b,i\}, \text{singletons}\}$, $\mathcal{E} = \{\mathcal{E}_i\}_{i\in I}$, and let $h \in G_X$ be given by

$$h(x) = x + 1, \text{ for all } x \in \mathbf{Z},\ h(b) = b\ .$$

Then $\{h^{-1}(C) \mid C \in \mathcal{E}_1\} = \{\{b,0\}, \text{singletons}\} \notin \mathcal{E}$. $\square$

For later reference we record the following simple observation concerning the relation ρ.

<u>PROPOSITION 3.1.7</u>. Let A be a well-separated subset of X (relative to some $CT(X,\mathcal{E},p,q)$).

(1) For $a \in A$ and $b \in X$ with $a \rho b$, the set $(A - \{a\}) \cup \{b\}$ is also well-separated. More generally:

(2) If $A_1 \subseteq A$ and B_1 is any (partial) ρ cross-section of the ρ-classes of elements in A_1, then $(A - A_1) \cup B_1$ is also well-separated. $\square$

A key technique used in the proof of Theorem 3.1.2, and indeed in much of the work of later sections, is the association of a certain order-automorphism H of C_q with each automorphism ϕ of S. Here C_q is regarded as a poset under set inclusion. This technique was used in the Baer-Levi case [8]. As a first step towards constructing H, we produce an order-automorphism H_p of C_p by looking at the restriction of ϕ to the minimal ideal CT(p,p). For functions in CT(p,p) there is a simple algebraic characterization of range inclusion:

<u>PROPOSITION 3.1.8.</u> Let $f,g \in CT(p,p)$. Then $R(f) \subseteq R(g)$ if and only if for each $k \in CT(p,p)$ such that $kg = g$ we have $kf = f$.

<u>Proof.</u> If $R(f) \subseteq R(g)$, then surely $kg = g$ implies $kf = f$.

For the converse, suppose $x \in R(f) - R(g)$. Choose a w.s. set $A \supseteq R(g)$ with $|A - R(g)| = p$. Fix $A \in \mathcal{E}$ and a cross-section $T = R(g) \dot{\cup} Y$ for A. Inasmuch as $g \in CT(p,p)$, $|Y| = p$. Write $A - R(g) = D_1 \dot{\cup} D_2$ where $|D_1| = |D_2| = p$ and $x \notin D_1$. Choose a bijection $k_1 : Y \to D_1$, and define $k : X \to X$ by declaring $\pi(k) = A$ and letting k act on the cross-section T as k_1 on Y and the identity on R(g). Then $k \in CT(p,p)$ because $R(k) = R(g) \cup D_1 \subseteq A$ and $|A - R(k)| = |D_2| = p$. Certainly $kg = g$, but $kf \neq f$ because $x \in R(f) - R(k)$. $\square$

<u>COROLLARY 3.1.9.</u> For $f,g \in CT(p,p)$ and $\phi \in \text{Aut } S$ we have:

(1) $R(f) \subseteq R(g)$ if and only if $R(\phi(f)) \subseteq R(\phi(g))$.

(2) $R(f) = R(g)$ if and only if $R(\phi(f)) = R(\phi(g))$.

<u>Proof.</u> Since CT(p,p) is the minimal ideal of S, ϕ induces, by its

restriction, an automorphism of $CT(p,p)$. (1) is now immediate from Proposition 3.1.8, while (2) is obvious from (1). □

In view of 3.1.9(2), ϕ gives rise in a natural way to a well-defined bijection $H_p : \mathcal{C}_p \to \mathcal{C}_p$, namely, given $A \in \mathcal{C}_p$

$$H_p(A) = R(\phi(f))$$

for any $f \in CT(p,p)$ with $R(f) = A$. Further, by 3.1.9(1), when $\mathcal{C}_p$ is regarded as a partially ordered set under set inclusion, we have:

PROPOSITION 3.1.10. H_p is an order-automorphism of $\mathcal{C}_p$. □

Although Corollary 3.1.9 will remain valid for general $f,g \in S = CT(X,\mathcal{E},p,q)$ when $W = X$, and hence ϕ will also induce an order-automorphism of $\mathcal{C}_q$, our method of establishing this is less direct. This is because we lack a simple algebraic characterization of range inclusion in the general case. The following lemmas actually apply to an arbitrary order-automorphism H of $\mathcal{C}_q$. Our immediate application will be to produce the order-automorphism of $\mathcal{C}_q$ associated with ϕ, using the already established H_p. However the results on the h_A maps are also used in later sections.

LEMMA 3.1.11. Given an order-automorphism $H : \mathcal{C}_q \to \mathcal{C}_q$, for each $A \in \mathcal{C}_q$ there is a unique map

$$h_A : A \to H(A)$$

such that

$$H(A - \{x\}) \cup \{h_A(x)\} = H(A) \qquad \forall x \in A\,.$$

Proof. Let $x \in A$. Observe that since $A - \{x\} \in \mathcal{C}_q$ and A covers $A - \{x\}$, $H(A)$ must cover $H(A - \{x\})$. Thus there is a uniquely determined element $h_A(x) \in H(A)$ such that $H(A) = H(A - \{x\}) \cup \{h_A(x)\}$. □

LEMMA 3.1.12. The map h_A in 3.1.11 is a bijection. In particular $h_A(A) = H(A)$.

Proof. Since H^{-1} is also an order-automorphism, there is by 3.1.11 a map $k_A : H(A) \to A$ such that

$$A = H^{-1}(H(A) - \{y\}) \cup \{k_A(y)\} \quad \forall y \in H(A).$$

Let $x \in A$. Then

$$\begin{aligned} A &= H^{-1}(H(A) - \{h_A(x)\}) \cup \{k_A h_A(x)\}, \quad \text{since } h_A(x) \in H(A) \\ &= H^{-1}(H(A - \{x\})) \cup \{k_A h_A(x)\} \\ &= (A - \{x\}) \cup \{k_A h_A(x)\} \end{aligned}$$

whence $k_A h_A(x) = x$. Similarly $h_A k_A(y) = y \quad \forall y \in H(A)$, showing h_A is a bijection with inverse k_A. □

LEMMA 3.1.13. For $A,B \in \mathcal{C}_q$ with $B \subseteq A$, the maps h_A and h_B agree on B.

Proof. Let $x \in B$. Now $B \subseteq A$ implies $H(B) \subseteq H(A)$ and so $h_B(x) \in H(A) = H(A - \{x\}) \cup \{h_A(x)\}$. If $h_B(x) \in H(A - \{x\})$, then $H(B) = H(B - \{x\}) \cup \{h_B(x)\} \subseteq H(A - \{x\})$, whence $B \subseteq A - \{x\}$, a contradiction. Thus $h_B(x) \in \{h_A(x)\}$ and hence $h_B(x) = h_A(x)$. □

LEMMA 3.1.14. For any well-separated set $U \subseteq X$, there exists a map $h : U \to W$ such that $h|_A = h_A$ for all $A \in \mathcal{C}_q$, $A \subseteq U$.

Proof. We simply define $h : U \to W$ by $h(x) = h_A(x)$ whenever $x \in A$, $A \in \mathcal{C}_q$, $A \subseteq U$. This is well-defined: for suppose also $x \in B$, $B \in \mathcal{C}_q$, $B \subseteq U$. Choose $\mathcal{C}_q$ sets $A_1 \subseteq A$, $B_1 \subseteq B$ such that $x \in A_1 \cap B_1$ and $A_1 \cup B_1 \in \mathcal{C}_q$. Let $C = A_1 \cup B_1$. By Lemma 3.1.13,

$$h_A(x) = h_{A_1}(x) = h_C(x) = h_{B_1}(x) = h_B(x). \quad □$$

PROPOSITION 3.1.15. Let ϕ be an automorphism of $S = CT(X,\mathcal{E},p,q)$ where $W = X$. Let $f,g \in S$. Then $R(f) \subseteq R(g)$ if and only if $R(\phi(f)) \subseteq R(\phi(g))$.

Proof. Let H_p be the order-automorphism of $\mathcal{C}_p$ given prior to 3.1.10. For $A \in \mathcal{C}_p$, let $h_A : A \to H_p(A)$ be as in Lemma 3.1.11 (with $p = q$ and $H = H_p$).

Observe that since $W = X$, for any $f \in S$

$$R(f) = \bigcup_{\ell \in CT(p,p)} R(f\ell)$$

and each $f\ell \in CT(p,p)$. Also, given arbitrary families $\{B_j\}$, $\{C_k\}$ of $\mathcal{C}_p$ sets lying inside some fixed $U \in \mathcal{C}_q$, we have

(3.1.1) $$\cup B_j \subseteq \cup C_k \quad \text{implies} \quad \cup H_p(B_j) \subseteq \cup H_p(C_k)$$

because by Lemmas 3.1.12 and 3.1.14

$$\begin{aligned} \cup B_j \subseteq \cup C_k \quad &\text{implies} \quad h(\cup B_j) \subseteq h(\cup C_k) \\ &\text{implies} \quad \cup h(B_j) \subseteq \cup h(C_k) \\ &\text{implies} \quad \cup H_p(B_j) \subseteq \cup H_p(C_k). \end{aligned}$$

Now suppose $R(f) \subseteq R(g)$. Notice that $\phi(CT(p,p)) = CT(p,p)$ because $CT(p,p)$ is the minimal ideal of S. Hence, letting ℓ and ℓ' range over $CT(p,p)$, we have by (3.1.1) and the definition of H_p that

$$R(f) = \bigcup_{\ell} R(f\ell) \subseteq R(g) = \bigcup_{\ell} R(g\ell)$$

$$\begin{aligned} \text{implies} \quad & \bigcup_{\ell} H_p(R(f\ell)) \subseteq \bigcup_{\ell} H_p(R(g\ell)) \\ \text{implies} \quad & \bigcup_{\ell} R(\phi(f\ell)) \subseteq \bigcup_{\ell} R(\phi(g\ell)) \\ \text{implies} \quad & \bigcup_{\ell} R(\phi(f)\phi(\ell)) \subseteq \bigcup_{\ell} R(\phi(g)\phi(\ell)) \\ \text{implies} \quad & \bigcup_{\ell'} R(\phi(f)\ell') \subseteq \bigcup_{\ell'} R(\phi(g)\ell') \\ \text{implies} \quad & R(\phi(f)) \subseteq R(\phi(g)). \end{aligned}$$

Similarly, by considering ϕ^{-1}, we have $R(\phi(f)) \subseteq R(\phi(g))$ implies $R(f) \subseteq R(g)$. □

By the same argument that we used for the H_p map, Proposition 3.1.15 ensures that the map $H : \mathcal{C}_q \to \mathcal{C}_q$ where, for $A \in \mathcal{C}_q$,

$$H(A) = R(\phi(f)) \text{ for any } f \in S \text{ with } R(f) = A,$$

is a well-defined order-automorphism of $\mathcal{C}_q$. Thus:

<u>PROPOSITION 3.1.16</u>. Given an automorphism ϕ of $S = CT(X,\&,p,q)$, where

$W = X$, there exists a unique order-automorphism $H : \mathcal{C}_q \to \mathcal{C}_q$ satisfying

$$H(R(f)) = R(\phi(f)) \quad \forall f \in S. \qquad \square$$

DEFINITION 3.1.17. We call H in 3.1.16 the *order-automorphism* of ϕ. $\square$

REMARK. When $W \neq X$ automorphisms need not have such an associated order-automorphism. This is shown in Section 3.5. $\square$

For the remainder of this section ϕ denotes a fixed automorphism of $S = CT(X,\mathcal{E},p,q)$, $W = X$, and H is the order-automorphism of ϕ. The following property of the h_A maps associated with H need not hold for a general order-automorphism of $\mathcal{C}_q$. It says that, although h_A and h_B need not agree on a common point $x \in A \cap B$, at least their values are ρ-equivalent.

PROPOSITION 3.1.18. Let H be the order-automorphism of ϕ. If $A,B \in \mathcal{C}_q$ and $x \in A \cap B$, then $h_A(x) \ \rho \ h_B(x)$.

Proof. Let $\mathcal{E}_i \in \mathcal{E}$. Choose $f \in S$ with

$$\pi(f) = \mathcal{E}_i, \ R(f) = A, \text{ and } f(x) = x.$$

Choose $\ell, g \in S$ such that $B = R(\ell)$ and $B - \{x\} = R(g)$. Using Lemma 3.1.11 and the definition of H, we have $H(B) - H(B - \{x\}) = \{h_B(x)\}$ and hence

$$R(\phi(\ell)) - R(\phi(g)) = \{h_B(x)\}. \tag{3.1.2}$$

Further, since f is 1-1 on any $\mathcal{C}_q$ set,

$$R(f\ell) - R(fg) = f(B) - f(B - \{x\}) = \{f(x)\} = \{x\}$$

whence by Lemma 3.1.13 we have

$$R(\phi(f\ell)) - R(\phi(fg)) = \{h_A(x)\}. \tag{3.1.3}$$

On the other hand, $R(\phi(f\ell)) - R(\phi(fg)) = R(\phi(f)\phi(\ell)) - R(\phi(f)\phi(g))$ $= \phi(f)(R(\phi(\ell)) - R(\phi(g))) = \{\phi(f)h_B(x)\}$ from (3.1.2). Combining this with (3.1.3) we have

(3.1.4) $$\phi(f)h_B(x) = h_A(x).$$

Since B is independent of A, we are free to interpret (3.1.4) in the case $B = A$. We then obtain

(3.1.5) $$\phi(f)h_A(x) = h_A(x).$$

Comparing (3.1.4) and (3.1.5) now gives $\phi(f)h_B(x) = \phi(f)h_A(x)$, whence

$$h_B(x) \;\&_j\; h_A(x)$$

for $\&_j = \pi(\phi(f)) \in \&$. But since ϕ is an onto map, $\&_j$ represents an arbitrary member of $\&$. Thus $h_B(x) \;\rho\; h_A(x)$. □

<u>REMARK</u>. If ϕ is inner, say induced by $h \in G_X$, then $h_A = h|_A$ so the h_A maps agree on common points. Conversely, as the proof of Theorem 3.1.2 will show, if the h_A maps agree on common points, then ϕ is inner and is induced by the common extension of the h_A maps. □

<u>COROLLARY 3.1.19</u>. If S is also reduced, there exists $h \in G_X$ such that $h(x) = h_A(x)$ for all $x \in A$ and $A \in C_q$.

<u>Proof</u>. The map $h : X \to X$ given by

$$h(x) = h_A(x), \quad \text{if } x \in A,\ A \in C_q$$

is now well-defined by Proposition 3.1.18 because $\rho = \iota_X$. By Lemma 3.1.11

$$H(A - \{x\}) \cup \{h(x)\} = H(A) \qquad \forall x \in A,\ A \in C_q\,.$$

Similarly by considering H^{-1}, which is the order-automorphism of ϕ^{-1}, there is a map $t : X \to X$ such that

$$H^{-1}(A - \{x\}) \cup \{t(x)\} = H^{-1}(A) \qquad \forall x \in A,\ A \in C_q\,.$$

Now

$$\begin{aligned} A &= H^{-1}(H(A)) \\ &= H^{-1}(H(A) - \{h(x)\}) \cup \{th(x)\} \\ &= H^{-1}(H(A - \{x\})) \cup \{th(x)\} \\ &= (A - \{x\}) \cup \{th(x)\} \end{aligned}$$

so $th(x) = x$. Similarly $ht(x) = x$, so h is a bijection with inverse t. $\square$

COROLLARY 3.1.20. For S and h as in 3.1.19,

$$H(A) = h(A), \text{ for all } A \in \mathcal{C}_q .$$

Proof. From Lemma 3.1.12 and Corollary 3.1.19,

$$H(A) = h_A(A) = h(A). \qquad \square$$

We are ready to return to:

Proof of Theorem 3.1.2.

Let $\phi \in \text{Aut } S$ where $S = CT(X,\mathscr{E},p,q)$ with $W = X$ and X reduced. Let H be the order-automorphism of ϕ, and let $h \in \mathcal{G}_X$ be as in Corollary 3.1.19. By Lemma 3.1.11 and Corollary 3.1.19, $H(A) - H(A - \{x\}) = \{h(x)\}$ for all $x \in A$, $A \in \mathcal{C}_q$. We show ϕ is induced by h.

Let $f \in S$. Let $x \in X$. Since $W = X$, we can choose $A \in \mathcal{C}_q$ such that $x \in A$, together with $\ell, g \in S$ such that $R(\ell) = A$, $R(g) = A - \{x\}$. Now $H(A) - H(A - \{x\}) = \{h(x)\}$ implies

$$R(\phi(\ell)) - R(\phi(g)) = \{h(x)\} .$$

Also $R(f\ell) - R(fg) = f(R(\ell) - R(g)) = \{f(x)\}$, hence

$$R(\phi(f\ell)) - R(\phi(fg)) = \{hf(x)\} .$$

On the other hand $R(\phi(f\ell)) - R(\phi(fg)) = R(\phi(f)\phi(\ell)) - R(\phi(f)\phi(g)) = \phi(f)(R(\phi(\ell)) - R(\phi(g))) = \{\phi(f)h(x)\}$, whence

$$\phi(f)h(x) = hf(x), \text{ for all } x \in X .$$

Thus $\phi(f) = hfh^{-1}$ for all $f \in S$. By Proposition 3.1.6, h is p.p. $\square$

REMARK 3.1.21. By Proposition 3.1.6, h is unique. $\square$

3.2. AUTOMORPHISMS WHEN W = X: GENERAL CASE

This section not only characterises the automorphisms of its title, but also establishes extensive groundwork required later. Consequently it is lengthy.

In the previous section, all automorphisms considered were inner. The main theorem of this section (Theorem 3.2.2) shows that all automorphisms of a Croisot-Teissier semigroup for which $W = X$ are locally inner, a concept made precise in Definition 3.2.31. We begin with an example of such an automorphism. Comments prior to Theorem 3.2.2, and the theorem itself, demonstrate that this example is the simplest available for which an outer automorphism exists.

EXAMPLE 3.2.1. Let $X = C \cup D$ where $|C| = |D| = p$ and $C \cap D = \{x,y\}$. For $c \in C$ and $d \in D$ with $c,d \neq x,y$ define a partition $\mathcal{E}_{c,d} = \{\{c,d\}, \{x,y\}, \text{singletons}\}$ on X and let $\mathcal{E} = \{\mathcal{E}_{c,d}\}$. Then $Y \subseteq X$ with $|Y| = p$ is well-separated if and only if Y contains at most one element of $\{x,y\}$ and is a subset of either C or D. Thus $X = W$ and since $X/\rho = \{\{x,y\}, \text{singletons}\}$, $S = CT(X,\mathcal{E},p,q)$ is not reduced.

Consider ϕ on S given by

$$\phi(f) = \begin{cases} (x,y)f & \text{for } R(f) \subseteq C, \\ f & \text{for } R(f) \subseteq D. \end{cases}$$

Certainly $\phi(f) \in S$ and ϕ is a bijection, while $\phi(fg) = (x,y)fg = \phi(f)\phi(g)$ when $R(f) \subseteq C$, and $\phi(fg) = fg = \phi(f)\phi(g)$ when $R(f) \subseteq D$, so ϕ is an automorphism of S. Suppose now that $\phi(f) = hfh^{-1}$ for all $f \in S$ and some bijection h of W. Take $A \subseteq C$ and $B \subseteq D$, each in C_q and containing x. Since $H(A) = h(A)$ where H is the order-automorphism of ϕ (see 3.1.17),

$h_A(x) = H(A) - H(A - \{x\}) = h(x)$, and similarly

$h_B(x) = h(x)$. By considering functions in S with ranges A and $A - \{x\}$, and recalling that $H(R(f)) = R(\phi(f))$, evidently $h_A(x) = y$. Similarly $h_B(x) = x$, a contradiction, so ϕ is outer. □

In this non-reduced example the local (h_A) descriptions of H fail to be restrictions of a common bijection h of W, so an outer automorphism results. Throughout this section, by means of a series of lemmas and propositions, we produce a canonical cover of X (which is independent of any automorphism), on the components of which these local descriptions agree. This yields a compatible system of bijections of W (see Definition 3.2.24) which serves to describe an automorphism. A key step will be the association with S of a reduced semigroup $\bar{S}$, and the use of Theorem 3.1.2.

<u>THEOREM 3.2.2.</u> Let X = W and let ϕ be an automorphism of $S = CT(X,\&,p,q)$. Then ϕ is locally inner, that is, for some compatible system $\{h_\alpha\}_{\alpha\in\Omega}$ of bijections of W,

(3.2.1) $$\phi(f) = h_\alpha f h_\alpha^{-1}, \text{ for all } f \in I_\alpha .$$

Conversely, given a compatible system $\{h_\alpha\}_{\alpha\in\Omega}$ of bijections of W, ϕ determined as in (3.2.1) is an automorphism of S. □

We begin the proof of this theorem by introducing $\bar{S}$, the reduced Croisot-Teissier semigroup associated with S, and show how it can be regarded as a factor semigroup of S. An automorphism on S will then give rise to an automorphism on $\bar{S}$, which is known to be inner.

<u>DEFINITION 3.2.3.</u> Let $\bar{W} = W/\rho$ and denote by $\bar{x}$ the ρ-equivalence class of $x \in W$. Let $\bar{\&}_i$ be the equivalence relation on $\bar{W}$ given by $\bar{x}\ \bar{\&}_i\ \bar{y}$ iff $x\ \&_i\ y$, for each $\bar{x},\bar{y} \in \bar{W}$. □

Note that since $\rho \subseteq \&_i$ for each i, the partition ρ induces is finer than that of any $\&_i$, and that the $\bar{\&}_i$- classes have the form $\{\bar{x} \mid x\ \&_i\ y\}$ for some fixed $y \in X$. The next three statements follow straightforwardly.

<u>LEMMA 3.2.4.</u> (1) $|\bar{W}| \geqslant p$

(2) $|\bar{W}/\bar{\&}_i| = |W/\&_i| = p$

(3) $Y \subseteq \bar{W}$ is w.s. iff $Y = \bar{A}$ for some w.s. $A \subseteq W$

$(\bar{A} = \{\bar{a} \mid a \in A\})$. □

These ensure that the following definition produces a non-empty Croisot-Teissier semigroup. Let $\bar{\mathcal{E}} = \{\bar{\mathcal{E}}_i\}_{i\in I}$.

<u>DEFINITION 3.2.5.</u> Given $S = CT(W,\mathcal{E},p,q)$, we term $\bar{S} = CT(\bar{W},\bar{\mathcal{E}},p,q)$ the *reduced semigroup associated with* S. □

It can be verified that $\bar{S}$ is reduced, that is $\cap\,\bar{\mathcal{E}}_i$ is the identity. Indeed, given $\bar{x},\bar{y}$ in $\bar{W}$,

$$\begin{aligned}
\bar{x} \cap \bar{\mathcal{E}}_i\, \bar{y} \quad &\text{iff} \quad \bar{x}\, \bar{\mathcal{E}}_i\, \bar{y} \quad \text{for each } i \in I \\
&\text{iff} \quad x\, \mathcal{E}_i\, y \quad \text{for each } i \in I \\
&\text{iff} \quad x\, \rho\, y, \quad \text{or } \bar{x} = \bar{y}\,.
\end{aligned}$$

We now introduce a key notion for the remainder of this section.

<u>DEFINITION 3.2.6.</u> A selfmap h of W is a *lifting* of the selfmap k of $\bar{W}$ if $\overline{h(x)} = k(\bar{x})$ for all $x \in W$. That is, the diagram

$$\begin{array}{ccc}
W & \xrightarrow{h} & W \\
\downarrow & & \downarrow \\
\bar{W} & \xrightarrow{k} & \bar{W}
\end{array}$$

commutes, where the vertical arrows represent the natural map. □

Define a map $\theta : S \to \bar{S}$ via $\theta(f) = \bar{f}$, where $\bar{f}(\bar{x}) = \overline{f(x)}$ for $\bar{x} \in \bar{W}$. Since $\pi(\bar{f}) = \overline{\pi(f)}$ and $R(\bar{f}) = \overline{R(f)} \in \bar{C}_q = \{\bar{A} \mid A \in C_q\}$, $\bar{f} \in \bar{S}$.

<u>LEMMA 3.2.7.</u> (1) θ is a homomorphism from S onto $\bar{S}$,

(2) $\ker\theta = \{(f,g) \mid \ell f = \ell g \text{ for all } \ell \in S\}$.

<u>Proof.</u> (1) Since $\overline{fg}(\bar{x}) = \overline{fg(x)} = \bar{f}(\overline{g(x)}) = \bar{f}(\bar{g}(\bar{x}))$, θ is a homomorphism. Given $k \in \bar{S}$, define a lifting $h \in S$ via $\overline{\pi(h)} = \pi(k)$ and $\overline{R(h)} = R(k)$. Then $\bar{h}(\bar{x}) = \overline{h(x)} = k(\bar{x})$ so $\bar{h} = k$ and θ is onto.

(2) Recall [7,p.22] that the kernel of the homomorphism θ is given by the following:

$$\ker\theta = \theta^{-1}\theta = \{(f,g) \in S \times S \mid \theta(f) = \theta(g)\}.$$

Now $(f,g) \in \ker\theta \Leftrightarrow \bar{f} = \bar{g} \Leftrightarrow \overline{f(x)} = \overline{g(x)}$ for all $x \in W \Leftrightarrow f(x)\ \rho\ g(x)$, for all $x \in W$. If $f(x)\ \rho\ g(x)$ for all $x \in W$, then $\ell f(x) = \ell g(x)$ for all $x \in W$ and $\ell \in S$, or $\ell f = \ell g$ for all $\ell \in S$. On the other hand, suppose $f(x) \not\rho\ g(x)$ for all x. Then there exists an $\mathcal{E}_i \in \mathcal{E}$ and an $x \in W$ such that $f(x) \not\mathcal{E}_i\ g(x)$. Choose $\ell \in S$ with $\pi(\ell) = \mathcal{E}_i$. Then $\ell f(x) \neq \ell g(x)$. □

Since $\ker\theta = \theta^{-1}\theta$, it is an equivalence on S. Moreover, $\ker\theta$ is a congruence on S, for if (f,g) and (f_1,g_1) are in $\ker\theta$, then

$$\theta(ff_1) = \theta(f)\theta(f_1) = \theta(g)\theta(g_1) = \theta(gg_1),$$

or $(ff_1,gg_1) \in \ker\theta$. Hence the following definition.

DEFINITION 3.2.8. Let μ be the congruence on S given by $f\ \mu\ g$ if and only if $\ell f = \ell g$ for all $\ell \in S$. □

The preceding lemma immediately provides the following statement.

PROPOSITION 3.2.9. $S/\mu \cong \bar{S}$. □

We return now to an automorphism ϕ of S. Such a ϕ permutes the μ-classes of S, since $\ell f = \ell g$ for all $\ell \in S$ iff $\ell\phi(f) = \ell\phi(g)$ for all $\ell \in S$. Thus $\bar{\phi}$ on $\bar{S}$ given by $\bar{\phi}(\bar{f}) = \overline{\phi(f)}$ is a well-defined automorphism of $\bar{S}$. Let H be the order-automorphism of ϕ, K that of $\bar{\phi}$ (Definition 3.1.17), and recall (Theorem 3.1.2) that there exists a unique p.p. $k \in \mathcal{G}_{\bar{W}}$ such that $\bar{\phi}(\bar{f}) = k\bar{f}k^{-1}$ for all $\bar{f} \in \bar{S}$. We have the following lemma linking H, K and k.

LEMMA 3.2.10. $\overline{H(A)} = K(\bar{A}) = k(\bar{A})$ for all $A \in \mathcal{C}_q$.

Proof. Choose $f \in S$ with $R(f) = A$, so $\bar{A} = R(\bar{f})$. Then, since

$H(A) = R(\phi(f))$ we have

$$\begin{aligned}
\overline{H(A)} &= \overline{R(\phi(f))} \\
&= R(\overline{\phi(f)}) && (\overline{R(g)} = R(\bar{g}) \text{ for all } g \in S) \\
&= R(\bar{\phi}(\bar{f})) && (\text{definition of } \bar{\phi}) \\
&= R(k\bar{f}k^{-1}) && (\bar{\phi} \text{ is inner}) \\
&= k(R(\bar{f})) \\
&= k(\bar{A}) = K(\bar{A}) && (\text{Corollary 3.1.20}). \qquad \square
\end{aligned}$$

That each h_A map associated with H (Lemma 3.1.11) is a local lifting of this k is the content of our next lemma.

<u>LEMMA 3.2.11</u>. $\overline{h_A(x)} = k(\bar{x})$ for all $x \in A \in C_q$.

<u>Proof</u>. Let $x \in A \in C_q$. Since A is well-separated we have $\bar{A} - \{\bar{x}\} = \overline{A - \{x\}}$ so

$$\begin{aligned}
K(\bar{A} - \{\bar{x}\}) \cup \{k(\bar{x})\} &= K(\bar{A}) && (\text{Lemma 3.2.10}) \\
&= \overline{H(A)} && (\text{Lemma 3.2.10}) \\
&= \overline{H(A - \{x\})} \cup \{\overline{h_A(x)}\} && (H(A) \text{ w.s. and Lemma 3.1.1} \\
&= K(\overline{A - \{x\}}) \cup \{\overline{h_A(x)}\} && (\text{Lemma 3.2.10}) \\
&= K(\bar{A} - \{\bar{x}\}) \cup \{\overline{h_A(x)}\},
\end{aligned}$$

whence $\overline{h_A(x)} = k(\bar{x})$. $\square$

The next proposition reveals that the $k \in G_{\bar{W}}$ associated with an automorphism ϕ of S can be lifted to a *bijection* of W. It is a suitably compatible collection of such liftings which will describe ϕ.

<u>PROPOSITION 3.2.12</u>. $|k(\bar{x})| = |\bar{x}|$ for all $x \in W$.

<u>Proof</u>. Let $x \in A \in C_q$ and $B = A - \{x\}$. Consider the map $m : \bar{x} \to k(\bar{x})$, defined for each $y \in \bar{x}$ by $m(y) = h_{B \cup \{y\}}(y)$ (note that $B \cup \{y\} \in C_q$ by Proposition 3.1.7(1)). That is, $H(B) \cup \{m(y)\} = H(B \cup \{y\})$.

Now $\overline{m(y)} = \overline{h_{B \cup \{y\}}(y)}$

$$= k(\bar{y}) \qquad \text{(Lemma 3.2.11)}$$
$$= k(\bar{x}),$$

so $m(y) \in k(\bar{x})$. Furthermore, m is one-to-one since H is one-to-one, so $|\bar{x}| \leqslant |k(\bar{x})|$. Also ϕ^{-1} on S yields $\bar{\phi}^{-1}$ on S, associated with k^{-1}, so by a similar argument

$$|k(\bar{x})| \leqslant |k^{-1}(k(\bar{x}))| = |\bar{x}| \quad \text{or} \quad |k(\bar{x})| = |\bar{x}| . \qquad \square$$

<u>COROLLARY 3.2.13.</u> $|\overline{h_A(x)}| = |\bar{x}|$, for all $x \in A \in C_q$.

<u>Proof.</u> This follows from Lemma 3.2.11 and Proposition 3.2.12. $\square$

In Example 3.2.1, $h_A(x) = y$ whereas $h_B(x) = x$. Hence in general $h_A(x) \neq h_B(x)$ when $A,B \in C_q$ and $x \in A \cap B$ (although always $h_A(x)\ \rho\ h_B(x)$). We next examine certain families M of C_q sets, independent of ϕ, such that $h_A(x) = h_B(x)$ whenever $A,B \in M$ and $x \in A \cap B$. These families turn out to be maximal with respect to possessing this property for all automorphisms ϕ (see Theorem 3.2.35 and its proof), although they are not defined this way.

The following discussion on finitely linked and λ-related C_q sets applies to all Croisot-Teissier semigroups CT(X,&,p,q), *even when* $W \neq X$.

<u>DEFINITION 3.2.14.</u> A pair of C_q sets A and B are *finitely linked through a* ρ*-class* U if there exist $F_1, \ldots, F_n \in C_q$ such that

$$F_1 = A,\ F_n = B,\ F_i \cap F_{i+1} \in C_q \quad \text{for } i = 1,\ldots, n-1$$
$$\text{and } F_i \cap U \neq \Phi \quad \text{for } i = 1,\ldots, n. \qquad \square$$

Our interest in linking through a ρ-class stems from:

<u>PROPOSITION 3.2.15.</u> Let $A,B \in C_q$ and suppose $x \in A \cap B$. If A and B are finitely linked through $\bar{x}$, then

$$h_A(x) = h_B(x) .$$

<u>Proof</u>. Suppose A and B are finitely linked through $\bar{x}$ via $F_1, \ldots, F_n \in C_q$. Observe that by Proposition 3.1.7(1) there is no loss of generality in assuming $x \in F_i$ for all i. By Lemma 3.1.13

$$\begin{aligned} h_A(x) &= h_{F_1}(x) = h_{F_1 \cap F_2}(x) = h_{F_2}(x) = h_{F_2 \cap F_3}(x) = \ldots \\ &= h_{F_n}(x) \\ &= h_B(x). \end{aligned}$$ □

It turns out that if $\bar{x}$ is a trivial (i.e. singleton) class, then always $h_A(x) = h_B(x)$. In anticipation of this we concentrate on linking through non-trivial ρ-classes.

<u>DEFINITION 3.2.16</u>. Let $A,B \in C_q$. We say A *is λ-related to* B (written A λ B) if A and B are finitely linked through U for each non-trivial ρ-class U which meets both A and B. □

By convention, A λ B automatically holds if A and B are disjoint or if only trivial ρ-classes meet both A and B. Although λ is a reflexive and a symmetric relation, it is not in general an equivalence. Indeed, choose in Example 3.2.1 C_q sets $A \subseteq C-D$ and $B \subseteq D-C$. Then

$$A \cup \{x\} \ \lambda \ B \quad \text{and} \quad B \ \lambda \ B \cup \{x\},$$

but $A \cup \{x\}$ and $B \cup \{x\}$ are not λ-related, so λ is not transitive.

Zorn's lemma ensures the existence of maximal families M of λ-related C_q sets.

<u>PROPOSITION 3.2.17</u>. Let M be a maximal family of λ-related C_q sets. Let $A,B \in C_q$, and A^c, B^c be their ρ-closures.

(1) If $B \in M$ and $A^c \subseteq B^c$, then $A \in M$.

(2) If $B \in M$ and $A \subseteq B$, then $A \in M$.

(3) $\bigcup_M A$ is a ρ-closed subset of W.

(4) $\{f \in S \mid R(f) \in M\}$ is a right ideal of S.

Proof. (1) It suffices to show that $\{A\} \cup \mathcal{M}$ is a λ-related family and then appeal to the maximality of $\mathcal{M}$ to deduce $A \in \mathcal{M}$. Let $C \in \mathcal{M}$ and suppose U is a non-trivial ρ-class meeting A and C. Since $A^c \subseteq B^c$, U meets B also. Moreover as $B \lambda C$, B and C are finitely linked through U, say via the $\mathcal{C}_q$ sets $F_1 = B, \ldots, F_n = C$. Write

$$A = D \dot{\cup} E \quad \text{with} \quad |D| = |E| \quad \text{and} \quad D \cap U \neq \Phi .$$

Set $G_1 = A$ and $G_2 = D \cup (E^c \cap B)$. Note that $|E^c \cap B| = |E|$ because $B^c \supseteq A^c \supseteq E^c$, and that $G_2 \in \mathcal{C}_q$ by Proposition 3.1.7(2). Thus G_1, G_2, $G_1 \cap G_2$ and $G_2 \cap F_1$ are $\mathcal{C}_q$ sets and G_1, G_2 meet U. Hence A is finitely linked to C through U via $G_1, G_2, F_1, F_2, \ldots, F_n$. This shows $A \lambda C$, which in turn shows $\{A\} \cup \mathcal{M}$ is a λ-related family.

(2) and (3) are immediate from (1).

(4) For $f,g \in S$, if $R(f) \in \mathcal{M}$ then $R(fg) \subseteq R(f)$ and so $R(fg) \in \mathcal{M}$ by (2). □

NOTATION 3.2.18. Let $\{\mathcal{M}_\alpha\}_{\alpha\in\Omega}$ be the collection of all maximal families of λ-related $\mathcal{C}_q$ sets. For each $\alpha \in \Omega$ let

$$A_\alpha = \bigcup_{\mathcal{M}_\alpha} A$$

$$I_\alpha = \{f \in S \mid R(f) \in \mathcal{M}_\alpha\} .$$ □

Since each $\mathcal{C}_q$ set is a member of some maximal family of λ-related $\mathcal{C}_q$ sets, by Proposition 3.2.17 we obtain the following decompositions of W and S, independently of any automorphism ϕ. These decompositions occupy a central position in the sequel.

PROPOSITION 3.2.19. Let $S = CT(X,\mathcal{E},p,q)$ be any Croisot-Teissier semigroup, and let W be the union of all well-separated subsets of X. Then

$$W = \bigcup_{\alpha\in\Omega} A_\alpha \qquad \text{(union of } \rho\text{-closed subsets of } X)$$

$$S = \bigcup_{\alpha\in\Omega} I_\alpha \qquad \text{(union of right ideals of } S). \qquad □$$

<u>REMARK</u>. The decomposition $S = \cup I_\alpha$ into right ideals, resulting from the above consideration of ranges of functions in S, is analogous to the decomposition $S = \cup T_i$ into left ideals, arising from consideration of partitions of functions in S. However, unlike the T_i, the I_α are not disjoint, and are never minimal. □

We return to the fixed automorphism $\phi : S \to S$ and its order-automorphism $H : \mathcal{C}_q \to \mathcal{C}_q$. Using the following lemma, we show ϕ permutes the I_α.

<u>LEMMA 3.2.20</u>. For $A,B \in \mathcal{C}_q$, if $A \lambda B$ then $H(A) \lambda H(B)$.

<u>Proof</u>. First observe that for $C \in \mathcal{C}_q$ and a ρ-class U, $U \cap C \neq \Phi$ if and only if $k(U) \cap H(C) \neq \Phi$: if $x \in U \cap C$ then by Lemma 3.2.11 $h_C(x) \in k(\bar{x}) = k(U)$, and by Lemma 3.1.11 $h_C(x) \in H(C)$, whence $h_C(x) \in k(U) \cap H(C)$. The converse follows by considering k^{-1} and H^{-1} (associated with ϕ^{-1}). Second, by Proposition 3.2.12 $|U| > 1$ if and only if $|k(U)| > 1$.

Now assume $A \lambda B$ and suppose V is a non-trivial ρ-class meeting $H(A)$ and $H(B)$. Write $V = k(U)$ for a ρ-class U. Then U is non-trivial and meets A and B. Hence $A \lambda B$, there exist $\mathcal{C}_q$ sets $F_1 = A$, $F_2, \ldots, F_n = B$ with $F_i \cap U \neq \Phi$ and $F_i \cap F_{i+1} \in \mathcal{C}_q$. Let $G_i = H(F_i)$ for $i = 1, \ldots, n$. By the first observation $G_i \cap V \neq \Phi$. Moreover $G_i \cap G_{i+1} \in \mathcal{C}_q$ because H is an order-automorphism of $\mathcal{C}_q$. Thus $H(A)$ and $H(B)$ are finitely linked through V via $G_1, \ldots, G_n$.
Hence $H(A) \lambda H(B)$. □

<u>PROPOSITION 3.2.21</u>. H permutes the $\mathcal{M}_\alpha$ while ϕ induces the corresponding permutation of the I_α.

<u>Proof</u>. Let $\mathcal{M}$ be a maximal family of λ-related $\mathcal{C}_q$ sets. By Lemma 3.2.20, $H(\mathcal{M}) = \{H(A) \mid A \in \mathcal{M}\}$ is a family of λ-related $\mathcal{C}_q$ sets. Suppose $B \in \mathcal{C}_q$ and $\{B\} \cup H(\mathcal{M})$ is a λ-related family. Write $B = H(A)$, $A \in \mathcal{C}_q$. By Lemma 3.2.20 applied to H^{-1}, $\{A\} \cup \mathcal{M}$ is a λ-related

family, so $A \in M$ by maximality of M. Thus $B \in H(M)$ and this shows that $H(M)$ is also maximal. Similarly $H^{-1}(M)$ is maximal. Hence the correspondence

$$M \mapsto H(M)$$

is a bijection of $\{M_\alpha\}_{\alpha\in\Omega}$ with inverse $M \mapsto H^{-1}(M)$.

Let $\alpha \in \Omega$ and suppose $H(M_\alpha) = M_\beta$. Using the definitions of H and the I_α, for $f \in S$ we have

$$\begin{aligned} f \in I_\alpha &\Leftrightarrow R(f) \in M_\alpha \\ &\Leftrightarrow H(R(f)) \in H(M_\alpha) = M_\beta \\ &\Leftrightarrow R(\phi(f)) \in M_\beta \\ &\Leftrightarrow \phi(f) \in I_\beta . \end{aligned}$$

Thus $\phi(I_\alpha) = I_\beta$ and hence ϕ induces a permutation of the I_α, mimicking the permutation of the M_α given by H. $\square$

Having produced the M_α we proceed to weld together the h_A maps for all A in a given M_α, and thus produce a common map on A_α.

<u>LEMMA 3.2.22.</u> Let $u : \Omega \to \Omega$ be the permutation of the index set Ω induced by H on $\{M_\alpha\}_{\alpha\in\Omega}$, or equivalently by ϕ on $\{I_\alpha\}_{\alpha\in\Omega}$. For each $\alpha \in \Omega$, there exists a unique bijection

$$h'_\alpha : A_\alpha \to A_{u(\alpha)}$$

such that

$$h'_{\alpha|A} = h_A \qquad \forall A \in M_\alpha .$$

<u>Proof.</u> Define $h'_\alpha : A_\alpha \to A_{u(\alpha)}$ by

$$h'_\alpha(x) = h_A(x)$$

whenever $x \in A$ and $A \in M_\alpha$. This map is well-defined: surely $h'_\alpha(x) \in A_{u(\alpha)}$ because $h_A(x) \in H(A)$ and $H(A) \in M_{u(\alpha)}$. Now suppose also $x \in B$ and $B \in M_\alpha$.

Case (i). Suppose $|\bar{x}| = 1$. By Corollary 3.2.13 $|\overline{h_A(x)}| = 1$, while by Proposition 3.1.18 $h_A(x)\ \rho\ h_B(x)$, whence $h_A(x) = h_B(x)$.

Case (ii). Suppose $|\bar{x}| > 1$. Then A and B are finitely linked through $\bar{x}$ because $A \lambda B$. By Proposition 3.2.15, $h_A(x) = h_B(x)$. Thus h'_α is well-defined.

Similarly by considering H^{-1} (which is the order automorphism of ϕ^{-1}), there is a map $t_\alpha : A_{u(\alpha)} \to A_\alpha$ such that

$$H^{-1}(B - \{x\}) \cup \{t_\alpha(x)\} = H^{-1}(B)$$

whenever $x \in B$ and $B \in M_{u(\alpha)}$. For $x \in A$ and $A \in M_\alpha$ we have

$$\begin{aligned} A &= H^{-1}(H(A)) \\ &= H^{-1}(H(A) - \{h'_\alpha(x)\}) \cup \{t_\alpha h'_\alpha(x)\} \\ &= H^{-1}(H(A - \{x\})) \cup \{t_\alpha h'_\alpha(x)\} \\ &= (A - \{x\}) \cup \{t_\alpha h'_\alpha(x)\} \end{aligned}$$

whence $t_\alpha h'_\alpha(x) = x$. Similarly $h'_\alpha t_\alpha(x) = x$ for $x \in A_{u(\alpha)}$, which shows h'_α is a bijection with inverse t_α. Clearly $h'_{\alpha|A} = h_A$ for all $A \in M_\alpha$ by the construction of h'_α while the uniqueness of h'_α is obvious. □

<u>PROPOSITION 3.2.23.</u> For each $\alpha \in \Omega$, there exists a bijection $h_\alpha : W \to W$ which is a lifting of k and satisfies

$$h_{\alpha|A} = h_A, \text{ for all } A \in M_\alpha .$$

<u>Proof.</u> For each ρ-class U which is disjoint from A_α, fix by Proposition 3.2.12 a bijection

$$s_U : U \to k(U).$$

Define $h_\alpha : W \to W$ by

$$h_\alpha(x) = \begin{cases} h'_\alpha(x) & \text{if } x \in A_\alpha , \\ s_U(x) & \text{if } x \notin A_\alpha,\ x \in U,\ U \in \bar{W} . \end{cases}$$

By Lemma 3.2.11 and Lemma 3.2.22, $\overline{h_\alpha(x)} = k(\bar{x})\quad \forall x \in A_\alpha$. Consequently, since $k : \bar{W} \to \bar{W}$ and $h'_\alpha : A_\alpha \to A_{u(\alpha)}$ are bijections, and A_α and $A_{u(\alpha)}$ are

ρ-closed by Proposition 3.2.17(3), it follows that h_α is a bijection. Clearly h_α lifts k. By Lemma 3.2.22 $h_{\alpha|A} = h'_{\alpha|A} = h_A$ for all $A \in M_\alpha$. □

DEFINITION 3.2.24. Term a set $\{g_\alpha\}_{\alpha\in\Omega} \subseteq G_W$ a *compatible system of bijections of* W if

(1) each g_α is a lifting of a common partition-preserving k in $G_{\bar{W}}$ (hence each g_α is partition-preserving), and

(2) $g_\alpha f = g_\beta f$, for all $f \in I_\alpha \cap I_\beta$, □

REMARK 3.2.25. Note that (1) is equivalent to each g_α being partition-preserving and $g_\alpha(x)\ \rho\ g_\beta(x)$ for all x in X, while (2) is equivalent to $g_{\alpha|A} = g_{\beta|A}$ for all $A \in M_\alpha \cap M_\beta$. □

The previous proposition indicates why such a melded family of liftings of a common p.p. bijection of $\bar{W}$ is crucial to the main theorem of this section: the $\{h_\alpha\}_{\alpha\in\Omega}$ of 3.2.23 is a compatible system, since for $A \in M_\alpha \cap M_\beta$,

$$h_{\alpha|A} = h_A = h_{\beta|A}.$$

Thus each automorphism determines a compatible system, and as our main theorem shows, is determined by such a system.

REMARK 3.2.26. For later reference note that each h_α^{-1}, $\alpha \in \Omega$, is a lifting of k^{-1}, so $h_\alpha^{-1}(x)\ \rho\ h_\beta^{-1}(x)$ and hence $h_\alpha^{-1}h_\beta(x)\ \rho\ x$, for all $x \in X$. □

LEMMA 3.2.27. A compatible system $\{g_\alpha\}_{\alpha\in\Omega}$ generates a permutation u of Ω such that for any $g \in \{g_\alpha\}_{\alpha\in\Omega}$, and for all $\alpha \in \Omega$,

$$g(A_\alpha) = A_{u(\alpha)}$$
$$g\, I_\alpha g^{-1} = I_{u(\alpha)}.$$

Proof. Since each $g \in \{g_\alpha\}_{\alpha\in\Omega}$ is partition-preserving, Proposition 3.1.6 implies that the map

$$\phi_g : S \to S, \text{ such that } f \mapsto gfg^{-1} \quad \forall f \in S$$

is an automorphism of S. Such a ϕ_g generates the order-automorphism

$$H : \mathcal{C}_q \to \mathcal{C}_q, \text{ such that } A \mapsto g(A) \quad \forall A \in \mathcal{C}_q ,$$

which permutes the M_α (Proposition 3.2.21). Hence the above generates a permutation

$$u_g : \Omega \to \Omega, \text{ such that } \alpha \mapsto \beta, \text{ where } M_\beta = \{g(A) \mid A \in M_\alpha\}, \text{ each } \alpha \in \Omega .$$

Let g_1 be any other element of $\{g_\alpha\}_{\alpha\in\Omega}$. By arguments similar to those above, g_1 generates a permutation u_{g_1} of Ω. We show that

$$u_g = u_{g_1} .$$

Indeed, let $\alpha \in \Omega$, $\beta = u_g(\alpha)$ and $\beta_1 = u_{g_1}(\alpha)$. Now,

$$B \in M_\beta \quad \text{iff} \quad B = g(A), \text{ for some } A \in M_\alpha .$$

Since $B = g(A) = g_1 g_1^{-1} g(A)$ and the σ-closure of $g_1^{-1} g(A)$ is just A, SEE ERRATA Proposition 3.2.17(1) ensures that $g_1^{-1} g(A) \in M_\alpha$. But then

$$B_1 \in M_{\beta_1} \quad \text{iff} \quad B_1 = g_1(A_1), \text{ some } A_1 \in M_\alpha$$

implies that $B \in M_{\beta_1}$. We deduce that

$$M_\beta = M_{\beta_1} \quad \text{and} \quad u_g = u_{g_1} = u .$$

Hence for each $\alpha \in \Omega$,

$$\begin{aligned} g(A_\alpha) &= g(\bigcup_{M_\alpha} A) \qquad \text{(Notation 3.2.18)} \\ &= \bigcup_{M_{u(\alpha)}} A \\ &= A_{u(\alpha)} . \end{aligned}$$

Since ϕ_g induces a permutation of I_α via

$$\phi_g : I_\alpha \to I_{u(\alpha)} \qquad \text{(Proposition 3.2.21)}$$

we conclude that

$$g\, I_\alpha g^{-1} = \phi_g(I_\alpha) = I_{u(\alpha)} . \qquad \square$$

LEMMA 3.2.28. Suppose that h and g in G_W are liftings of the partition-preserving $k \in G_{\bar{W}}$. Then for each $A \in C_q$, $h(A) = g(A)$ if and only if $h_{|A} = g_{|A}$.

Proof. The sufficiency is clear, so suppose $a \in A \in C_q$ and $h(A) = g(A)$. Then $\overline{h(a)} = k(\bar{a}) = \overline{g(a)}$ or $h(a)\ \rho\ g(a)$. But $h(A) = g(A)$ is well-separated (h and g are p.p.) so $h(a) = g(a)$. □

REMARK 3.2.29. Let $\{g_\alpha\}_{\alpha\in\Omega} \subseteq G_W$ be liftings of a common p.p. $k \in G_{\bar{W}}$ and u be the associated permutation of Ω. Then Lemma 3.2.28 enables us to show that $\{g_\alpha\}_{\alpha\in\Omega}$ is a compatible system if and only if each $H_\alpha : M_\alpha \to M_{u(\alpha)}$, which sends $A \in C_q$ to $\dot{g}_\alpha(A)$, is the restriction of a common map H of C_q. Indeed, if $\{g_\alpha\}_{\alpha\in\Omega}$ is a compatible system, then $g_{\alpha|A} = g_{\beta|A}$, for all $A \in M_\alpha \cap M_\beta$ (Remark 3.2.25) or $g_\alpha(A) = g_\beta(A)$ (Lemma 3.2.28). Hence we define a map H of C_q via $H(A) = H_\alpha(A)$, $A \in C_q$.

For the converse, if each H_α is a restriction of a common map H of C_q, then

$$g_\alpha(A) = H_\alpha(A) = H(A) = H_\beta(A) = g_\beta(A),$$

for all $A \in M_\alpha \cap M_\beta$, or, by Lemma 3.2.28, $g_{\alpha|A} = g_{\beta|A}$, so $\{g_\alpha\}_{\alpha\in\Omega}$ is a compatible system (Remark 3.2.25).

For the $\{h_\alpha\}$ of 3.2.23 this common map H is the order-automorphism of ϕ. □

Certainly both ϕ and ϕ^{-1} determine compatible systems. In the converse of the main theorem we show that a compatible system determines an automorphism, but our method requires that we produce the "inverse" compatible system directly from the given one. To this end we need:

LEMMA 3.2.30. If $\{g_\alpha\}_{\alpha\in\Omega}$ is a compatible system, then so too is $\left\{g^{-1}_{u^{-1}(\alpha)}\right\}_{\alpha\in\Omega}$.

Proof. Since g_α lifts k, g_α^{-1} lifts k^{-1} for each $\alpha \in \Omega$. Now observe that if $C, D \in M_\gamma \cap M_\delta$, $\gamma, \delta \in \Omega$, and $g_\gamma(C) = g_\delta(D)$, then $C = D$. This follows because $g_\gamma(C) = g_\delta(D) = g_\gamma(D)$ and g_γ is a bijection. For $A \in M_\alpha \cap M_\beta$,

$$g_{u^{-1}(\alpha)}\left(g^{-1}_{u^{-1}(\alpha)}\quad(A)\right) = g_{u^{-1}(\beta)}\left(g^{-1}_{u^{-1}(\beta)}\quad(A)\right).$$

Certainly $g^{-1}_{u^{-1}(\alpha)}$ (A) is in $M_{u^{-1}(\alpha)}$, while it also lies in $M_{u^{-1}(\beta)}$, since $\left(g^{-1}_{u^{-1}(\alpha)}\ (A)\right)^c = \left(g^{-1}_{u^{-1}(\beta)}\ (A)\right)^c$ and Proposition 3.2.17(1) provides that such equivalent C_q sets lie in the same maximal families. Thus $g^{-1}_{u^{-1}(\alpha)}$ (A) = $g^{-1}_{u^{-1}(\beta)}$ (A), from our first observation, or $g^{-1}_{u^{-1}(\alpha)}\Big|_A = g^{-1}_{u^{-1}(\beta)}\Big|_A$, by Lemma 3.2.28. □

DEFINITION 3.2.31. An automorphism ϕ of S is *locally inner* if there exists a compatible system of bijections $\{h_\alpha\}_{\alpha\in\Omega}$ of W such that $\phi(f) = h_\alpha f h_\alpha^{-1}$ for all $f \in I_\alpha$. □

Recall that S is covered by the right ideals, $\{I_\alpha\}_{\alpha\in\Omega}$. To say that ϕ is locally inner means that on each I_α, ϕ agrees with the inner automorphism of S induced by the p.p. bijection h_α (see 3.1.6).

Proof of Theorem 3.2.2. Let ϕ be an automorphism of S, and $\{h_\alpha\}_{\alpha\in\Omega}$ an associated compatible system (Proposition 3.2.23). We show $\phi(f) = h_\alpha f h_\alpha^{-1}$ for $f \in I_\alpha$. Take $x \in W$, say $x \in A_\beta$. It is possible to find $g, \ell \in I_\beta$ such that $R(\ell) - R(g) = \{x\}$, since M_β is closed under C_q-subset formation (Proposition 3.2.17(2)). By 3.1.11, the definition of H, and the fact that $h_{\beta|A} = h_A$ for $A \in M_\beta$, we have

$$R(\phi(\ell)) - R(\phi(g)) = \{h_\beta(x)\}.$$

On the other hand, $R(f\ell) - R(fg) = \{f(x)\}$, and since $f\ell, fg \in I_\alpha$, again by the above argument,

$$R(\phi(f\ell)) - R(\phi(fg)) = \{h_\alpha f(x)\}.$$

It follows that

$\{h_\alpha f(x)\} = \phi(f)[R(\phi(\ell)) - R(\phi(g))]$, since $\phi(f)$ is one-to-one on $R(\phi(\ell))$, or $h_\alpha f(x) = \phi(f)h_\beta(x)$. Now by Remark 3.2.25, $h_\beta(x) \;\rho\; h_\alpha(x)$, so $\phi(f)h_\alpha(x) = \phi(f)h_\beta(x) = h_\alpha f(x)$ for all $x \in X$, or $\phi(f) = h_\alpha f h_\alpha^{-1}$ for $f \in I_\alpha$.

Conversely, let $\{h_\alpha\}_{\alpha\in\Omega}$ be a compatible system, involving k (see Definition 3.2.24) and the permutation u of Ω (Lemma 3.2.27). Define $\phi(f) = h_\alpha f h_\alpha^{-1}$ for all $f \in I_\alpha$. Since h_α is p.p., $\phi(f) \in S$. In order to show ϕ is well-defined, suppose $f \in I_\alpha \cap I_\beta$. Then

$$\begin{aligned} h_\alpha f h_\alpha^{-1} &= h_\beta f h_\alpha^{-1} &&, \text{ since } \{h_\alpha\}_{\alpha\in\Omega} \text{ is a compatible system,} \\ &= h_\beta f h_\beta^{-1} &&, \text{ since } h_\alpha^{-1}(x) \;\rho\; h_\beta^{-1}(x) \quad \text{(Remark 3.2.26).} \end{aligned}$$

We take $f \in I_\alpha$, $g \in I_\beta$ and show that ϕ is a homomorphism. Now

$$\begin{aligned} \phi(f)\phi(g) &= (h_\alpha f h_\alpha^{-1})(h_\beta g h_\beta^{-1}) \\ &= h_\alpha (f h_\alpha^{-1} h_\beta)(g h_\beta^{-1}) \\ &= h_\alpha f (g h_\beta^{-1}) &&, \text{ since } h_\alpha^{-1} h_\beta(x) \;\rho\; x \quad \text{(Remark 3.2.26)} \\ &= h_\alpha f g h_\alpha^{-1} &&, \text{ since } h_\alpha^{-1}(x) \;\rho\; h_\beta^{-1}(x) \\ &= \phi(fg) &&, \text{ since } fg \in I_\alpha . \end{aligned}$$

In a similar way, χ given by

$$\chi(f) = h^{-1}_{u^{-1}(\alpha)} f h_{u^{-1}(\alpha)} \quad \text{for } f \in I_\alpha ,$$

is a homomorphism of S, since $\{h^{-1}_{u^{-1}(\alpha)}\}_{\alpha\in\Omega}$ is a compatible system (Lemma 3.2.30) involving k^{-1}, which in turn induces u^{-1} on Ω.

We show that χ is the inverse of ϕ. For $f \in I_\alpha$,

$$\begin{aligned} \chi\phi(f) &= \chi(h_\alpha f h_\alpha^{-1}) \\ &= h^{-1}_{u^{-1}u(\alpha)} h_\alpha f h_\alpha^{-1} h_{u^{-1}u(\alpha)} &&, \text{ since by Lemma 3.2.27 } h_\alpha f h_\alpha^{-1} \in I_{u(\alpha)} \\ &= h_\alpha^{-1} h_\alpha f h_\alpha^{-1} h_\alpha &&, \text{ since u is a bijection} \\ &= f. \end{aligned}$$

Also,

$$\phi\chi(f) = \phi(h^{-1}_{u^{-1}(\alpha)} fh_{u^{-1}(\alpha)})$$

$$= h_{u^{-1}(\alpha)} h^{-1}_{u^{-1}(\alpha)} fh_{u^{-1}(\alpha)} h^{-1}_{u^{-1}(\alpha)} \quad , \text{ since } h^{-1}_{u^{-1}(\alpha)} fh_{u^{-1}(\alpha)} \in I_{u^{-1}(\alpha)}$$

by Lemma 3.2.27 again

$$= f.$$

We conclude that ϕ is a bijection, and hence is an automorphism of S. □

REMARK 3.2.32. Given $\phi \in \text{Aut } S$ we term a compatible system $\{h_\alpha\}_{\alpha\in\Omega}$ satisfying

$$\phi(f) = h_\alpha f h_\alpha^{-1}, \text{ for all } f \in I_\alpha$$

a *compatible system associated with* ϕ. Such a system is unique to the following extent: if $\{g_\alpha\}_{\alpha\in\Omega}$ is another such system, then $g_\alpha(x) \;\rho\; h_\alpha(x)$ for all $x \in W$ and $g_\alpha(x) = h_\alpha(x)$ for all $x \in A_\alpha$. For since both $\bar{g}_\alpha$ and $\bar{h}_\alpha$ induce $\bar{\phi}$, by the uniqueness of k, $\bar{g}_\alpha = k = \bar{h}_\alpha$ so $g_\alpha(x) \;\rho\; h_\alpha(x)$. Now let $x \in A_\alpha$, say $x \in A$ for $A \in M_\alpha$, and $B = A - \{x\}$. By 3.2.17(2), $B \in M_\alpha$. Choose $f,g \in S$ with $R(f) = A$, $R(g) = B$. Then $f,g \in I_\alpha$ so $h_\alpha(R(f) - R(g)) = R(\phi(f)) - R(\phi(g))$ or $\{h_\alpha(x)\} = H(A) - H(B) = \{h_A(x)\}$, whence $h_\alpha(x) = h_A(x)$. Similarly $g_\alpha(x) = h_A(x)$, so $g_\alpha(x) = h_\alpha(x)$. □

COROLLARY 3.2.33. Let $S = CT(X,\mathcal{E},p,q)$ and $X = W$. Then any one of the following structures determines any other:

(1) an automorphism ϕ of S,

(2) a compatible system of bijections $\{h_\alpha\}_{\alpha\in\Omega}$ of W,

(3) an order-automorphism H of C_q, together with a partition-preserving bijection k of $\bar{W}$ such that $\overline{H(A)} = k(\bar{A})$ for all $A \in C_q$.

Proof. (1) ⇒ (3): Certainly ϕ determines the order-automorphism H (Proposition 3.1.16) while $\bar{\phi}$ yields k (Theorem 3.1.2) for which $\overline{H(A)} = k(\bar{A})$ for all $A \in C_q$ (Lemma 3.2.10).

(3) $\Rightarrow$ (2): The partial h_A maps can be derived as in 3.1.11 (with the properties of 3.1.12, 3.1.13 and 3.1.14). Then the machinery for producing the compatible system $\{h_\alpha\}_{\alpha\in\Omega}$, namely 3.2.11, 3.2.12, 3.2.13, 3.2.15, 3.2.22 and 3.2.23, operates as before provided we can show $h_A(x)\ \rho\ h_B(x)$ for all $A,B \in \mathcal{C}_q$ and $x \in A \cap B$, and $\overline{H^{-1}(A)} = k^{-1}(\bar{A})$ for all $A \in \mathcal{C}_q$. But

$$\begin{aligned}
\{\overline{h_A(x)}\} &= \overline{H(A) - H(A - \{x\})} \\
&= \overline{H(A)} - \overline{H(A - \{x\})} && \text{(A is w.s. and } x \in A) \\
&= k(\bar{A}) - k(\overline{A - \{x\}}) && \text{(given property)} \\
&= k(\bar{A}) - k(\bar{A} - \{\bar{x}\}) && (A - \{x\} \text{ is w.s.}) \\
&= \{k(\bar{x})\}.
\end{aligned}$$

Similarly $\overline{h_B(x)} = k(\bar{x})$, so $h_B(x)\ \rho\ h_A(x)$.

Inserting $H^{-1}(A)$ in $\overline{H(A)} = k(\bar{A})$, for A a $\mathcal{C}_q$ set, gives $\overline{HH^{-1}(A)} = k(\overline{H^{-1}(A)})$ or $\overline{H^{-1}(A)} = k^{-1}(\bar{A})$.

(2) $\Rightarrow$ (1): This is the content of the converse of Theorem 3.2.2. □

<u>REMARK.</u> It is easy to produce an order-automorphism H of $\mathcal{C}_q$ which is not associated with an automorphism ϕ of S, even if the h_A maps satisfy $h_A(x)\ \rho\ h_B(x)$ for $x \in A \cap B$. For example, fix a,b and c in X and let

$$\mathcal{E}_1 = \iota_X,\ \mathcal{E}_2 = \{\{a,b\}, \text{singletons}\},\ \mathcal{E}_3 = \{\{a,b,c\}, \text{singletons}\}$$

and $\mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$. Then $Y \subseteq X$ with $|Y| = |X|$ is w.s. if and only if Y contains at most one element of $U = \{a,b,c\}$. Thus $X = W$ and $\rho = \iota_X$. Let $h = (b,c)$. This h is not p.p., for

$$\{h(E) \mid E \in \mathcal{E}_2\} = \{\{a,c\}, \text{singletons}\} \notin \mathcal{E}\ .$$

However, $h(U) = U$, so for any $Y \subseteq X$

$$|Y \cap U| = |h(Y \cap U)| = |h(Y) \cap h(U)| = |h(Y) \cap U|\ ,$$

and so

$$Y \text{ is w.s.} \Leftrightarrow |Y \cap U| \leqslant 1 \Leftrightarrow |h(Y) \cap U| \leqslant 1 \Leftrightarrow h(Y) \text{ is w.s.}$$

Hence also Y is w.s. iff $h^{-1}(Y)$ is w.s., that is h and h^{-1} preserve w.s. sets. Now take $H(A) = h(A)$ for all $A \in \mathcal{C}_q$. Since h is not p.p., H is not associated with any automorphism ϕ of S. Thus k in (3) (Corollary 3.2.33) is essential. □

EXAMPLE 3.2.1. (revisited) The example with which we began this section usefully illustrates the mechanism of Theorem 3.2.2. Recall $X = W = C \cup D$ with $|C| = |D| = p$ and $C \cap D = \{x,y\} = U$, say. Also $\mathcal{E} = \{\mathcal{E}_{c,d}\}$ where $c \in C - U$, $d \in D - U$ and $\mathcal{E}_{c,d}$ links c to d, and x to y. As pointed out earlier, only one non-trivial ρ-class, U, exists, while $\mathcal{C}_q = \{A \subseteq C \text{ or } D \mid |A| = p,\ |A \cap U| \leqslant 1, \text{ and } (A \subseteq C \text{ say})\ |C - A| \geqslant q\}$. For $\mathcal{C}_q$ sets meeting U it is evident that they are finitely linked through U if and only if either they both lie in C, or both lie in D. Thus there are precisely two maximal families, M_1 and M_2, of λ-related sets, namely

$$M_1 = \{A \in \mathcal{C}_q \mid A \subseteq C \text{ or } A \subseteq D - U\} \quad \text{and}$$
$$M_2 = \{A \in \mathcal{C}_q \mid A \subseteq D \text{ or } A \subseteq C - U\}.$$

So $\Omega = \{1,2\}$ and $S = I_1 \cup I_2$.

The partition-preserving bijections of W are those which fix U (set-wise) and either fix or exchange $C - U$ and $D - U$. Note that $A_1 = A_2 = W$ so by Remark 3.2.32, automorphisms here are in one-to-one correspondence with compatible systems. Such a system is a pair $\{h_1,h_2\}$ of bijections of W which are p.p. and agree on $W - U$. Thus by Theorem 3.2.2 the automorphisms of S simplify to the form

$$\phi(f) = \begin{cases} h_1 f h_1^{-1} & \text{if } R(f) \subseteq C, \\ h_2 f h_2^{-1} & \text{if } R(f) \subseteq D, \end{cases}$$

where $\{h_1,h_2\}$ is a compatible system. Since $W = A_1 = A_2$, Remark 3.2.32 gives that ϕ is outer if and only if $h_1(x) \neq h_2(x)$. For this reason our earlier example with $h_1 = (x,y)$, and h_2 the identity on W is the simplest available.

For this example we describe the structure of Aut S and Inn S. Let $A_2 = \{\iota_S, \tau_0\}$ be the subgroup of Aut S, where

$$\tau_0(f) = \begin{cases} (x,y)f & \text{if } R(f) \subseteq C, \\ f & \text{if } R(f) \subseteq D, \end{cases}$$

already described in Example 3.2.1. We show that

$$\text{(3.2.2)} \qquad \text{Aut } S \cong \text{Inn } S \times A_2 .$$

Indeed, let $\chi : \text{Aut } S \to \text{Inn } S \times A_2$ be given by

$$\chi(\phi) = (\tau\phi, \tau) ,$$

where $\tau \in A_2$ is such that $\tau\phi \in \text{Inn } S$. Since for $\tau \in A_2$ and $\phi \in \text{Aut } S$, $\tau\phi \in \text{Inn } S$ if and only if either $\tau = \iota_S$ and $\phi \in \text{Inn } S$ or $\tau = \tau_0$ and $\phi \notin \text{Inn } S$, χ is well-defined. Also, the map $\xi : \text{Inn } S \times A_2 \to \text{Aut } S$ given by $\xi((\phi,\tau)) = \tau\phi$, $\phi \in \text{Inn } S$ and $\tau \in A_2$ is the inverse of χ. Indeed, for a $\phi \in \text{Aut } S$

$$\xi\chi(\phi) = \xi((\tau\phi,\tau)) = \tau^2\phi = \phi ,$$

where $\tau \in A_2$ is such that $\tau\phi \in \text{Inn } S$. Similarly, for $(\phi,\tau) \in \text{Inn } S \times A_2$,

$$\chi\xi((\phi,\tau)) = \chi(\tau\phi) = (\tau^2\phi,\tau) = (\phi,\tau) .$$

To show that χ is an isomorphism, take $\phi,\phi' \in \text{Aut } S$, then

$$\chi(\phi)\chi(\phi') = (\tau\phi,\tau)(\tau'\phi',\tau')$$

where $\tau,\tau' \in A_2$ are such that $\tau\phi, \tau'\phi' \in \text{Inn } S$. Since for any τ in A_2 and ϕ in Aut S, $\tau\phi = \phi\tau$, we can write:

$$\begin{aligned} \chi(\phi)\chi(\phi') &= (\tau\phi\tau'\phi', \tau\tau') \\ &= (\tau\tau'\phi\phi', \tau\tau'). \end{aligned}$$

Here $\tau\tau' \in A_2$ is such that $\tau\tau'\phi\phi' = \tau\phi\tau'\phi' \in \text{Inn } S$, so that

$$\chi(\phi)\chi(\phi') = (\tau\tau'\phi\phi', \tau\tau') = \chi(\phi\phi').$$

Let $C_2 = \{\iota, \beta\}$ be the cyclic group of order 2. We can rewrite (3.2.2) as

$$\mathrm{Aut}\ S \cong \mathrm{Inn}\ S \times C_2 .$$

Denote $C - U$ and $D - U$ by E and F respectively. We show that

$$\mathrm{Inn}\ S \cong \mathcal{G}_E \wr C_2 \times C_2 ,$$

where $\mathcal{G}_E \wr C_2$ is the wreath product of $\mathcal{G}_E$ and C_2. That is the group comprising all (δ, f_1, f_2), where $\delta \in C_2$, $f_1, f_2 \in \mathcal{G}_E$ with the product of (δ, f_1, f_2) and (δ', f_1', f_2') is given by $(\delta\delta', f_1 f_1', f_2 f_2')$ if $\delta' = \iota$ and by $(\delta\delta', f_2 f_1', f_1 f_2')$ if $\delta' = \beta$. Let $H = \{h \in \mathcal{G}_X \mid \phi_h \in \mathrm{Inn}\ S\}$. Clearly

$$H \cong \mathrm{Inn}\ S .$$

SEE ERRATA

Choose a bijection $g : E \to F$ and let

$$\psi : H \to \mathcal{G}_E \wr C_2 \times C_2$$

be given as follows:

$$\text{(3.2.3)} \qquad \psi(h) = \begin{cases} (\iota, h|_E,\ g^{-1}h|_F g, \gamma), & \text{if } h(E) = E, \\ (\beta, g^{-1}h|_E,\ h|_F g, \gamma), & \text{if } h(E) = F, \end{cases}$$

where $\gamma = \iota$ iff $h|_U = \iota_U$.

We show that ψ is an isomorphism. Let $\xi : \mathcal{G}_E \wr C_2 \times C_2 \to H$ be given by $\xi(\delta, f_1, f_2, \gamma) = h$ such that if $\delta = \iota$, then $h|_E = f_1$, $h|_F = g f_2 g^{-1}$; if $\delta = \beta$, then $h|_E = g f_1$, $h|_F = f_2 g^{-1}$ and $h|_U = \iota_U$ iff $\gamma = \iota$. Then for $h \in H$ with $h(E) = E$

$$\xi\psi(h) = \xi(\iota, h|_E,\ g^{-1}h|_F g, \gamma) = h'$$

with $h'|_E = h|_E$; $h'|_F = g g^{-1} h|_F g g^{-1} = h|_F$ and $h'|_U = h|_U$. Thus $h = h'$. Similarly for $h \in H$ with $h(E) = F$, $\xi\psi(h) = h$. Analogously we can show that $\psi\xi$ is the identity. We conclude that ξ is the inverse of ψ and ψ is a bijection.

Finally we show that ψ is a homomorphism. Let $h, h' \in H$. Consider the following two cases:

1. $h'(E) = E$.

We have that

$$\psi(h)\psi(h') = (\delta, f_1, f_2, \gamma)(\iota, h'|_E, g^{-1}h'|_F g, \gamma'),$$

where $\delta, f_1, f_2, \gamma, \gamma'$ are given according to (3.2.3). Then

$$\psi(h)\psi(h') = (\delta, f_1 h'|_E, f_2 g^{-1}h'|_F g, \gamma\gamma').$$

If $h(E) = E$, then

$$\begin{aligned}\psi(h)\psi(h') &= (\iota, h|_E h'|_E, g^{-1}h|_F g g^{-1}h'|_F g, \gamma\gamma')\\ &= (\iota, hh'|_E, g^{-1}hh'|_F g, \gamma\gamma')\\ &= \psi(hh').\end{aligned}$$

If $h(E) = F$, then

$$\begin{aligned}\psi(h)\psi(h') &= (\beta, g^{-1}h|_E h'|_E, h|_F g g^{-1}h'|_F g, \gamma\gamma')\\ &= (\beta, g^{-1}hh'|_E, hh'|_F g, \gamma\gamma')\\ &= \psi(hh').\end{aligned}$$

2. $h'(E) = F$.

We have as above

$$\begin{aligned}\psi(h)\psi(h') &= (\delta, f_1, f_2, \gamma)(\beta, g^{-1}h'|_E, h'|_F g, \gamma')\\ &= (\delta\beta, f_2 g^{-1}h'|_E, f_1 h'|_F g, \gamma\gamma').\end{aligned}$$

As in Case 1 we can check that the latter equals to $\psi(hh')$. □

We lead up now to a theorem (see 3.2.35) which immediately reveals that outer automorphisms exist in this example. It shows that it is the existence of more than one maximal λ-related family, rather than a non-trivial ρ-class, which ensures outer automorphisms are present.

LEMMA 3.2.34. Suppose $e : W \to W$ is a bijection such that $e(x)\ \rho\ x$ for all $x \in W$, and J is a right ideal of $S = CT(W, \mathcal{E}, p, q)$ satisfying:

(a) $eJ = J$

(b) For $f, g \in S$, if $f \notin J$ but $fg \in J$, then $e(fg) = fg$.

Then: (1) The map $\phi : S \to S$ given by

$$\phi(f) = \begin{cases} f & \text{if } f \in J \\ ef & \text{if } f \notin J \end{cases}$$

is an automorphism.

(2) ϕ is outer if there exist $x,y \in W$ such that

$$e(x) = y \neq x$$
$$x \in R(f) \text{ for some } f \notin J$$
$$y \in R(g) \text{ for some } g \in J.$$

Proof. (1) To show ϕ is a homomorphism, first observe that $f\phi(g) = fg \quad \forall f,g \in S$ because $\phi(g)(x)\ \rho\ g(x)$. Hence for $f \in J$,

$$\phi(fg) = fg = f\phi(g) = \phi(f)\phi(g).$$

Suppose $f \notin J$. If $fg \in J$ then by (b)

$$\phi(fg) = fg = e(fg) = (ef)\phi(g) = \phi(f)\phi(g).$$

On the other hand if $fg \notin J$

$$\phi(fg) = e(fg) = (ef)\phi(g) = \phi(f)\phi(g).$$

ϕ is a bijection because it has an inverse given by $\phi^{-1}(f) = f$ for $f \in J$, $\phi^{-1}(f) = e^{-1}f$ for $f \notin J$.

(2) Assume x and y exist with the three properties.

Claim: (i) *We can choose* $f \notin J$ *with* $f(x) = x$.

For suppose $x = f(w)$ for some $w \in W$. Since S acts transitively on W, we can choose $f_1 \in S$ with $f_1(x) = w$. Then $ff_1(x) = f(w) = x$. If $ff_1 \in J$ then by (b), $e(ff_1) = ff_1$ implies $e(x) = x$, a contradiction. Now replace f by ff_1 to obtain (i).

Suppose by way of a contradiction that ϕ is inner, say for some $h \in G_W$, $\phi(t) = hth^{-1}$ for all $t \in S$.

Claim: (ii) *There exists* $\ell \in J$ *with* $\ell(x) = y$ *but* $\ell h^{-1}(x) \neq fh^{-1}(x)$.

By hypothesis there exists $g \in J$ with $y \in R(g)$. Let $v = fh^{-1}(x)$. Observe $y \neq v$, otherwise both x and y are in the w.s. set $R(f)$, contrary to $x\ \rho\ y$. Choose $g_1 \in S$ with $y \in R(gg_1)$ but $v \notin R(gg_1)$. Let $g_2 = gg_1$ and suppose $g_2(u) = y$ for $u \in W$. Choose $g_3 \in S$ with $g_3(x) = u$. Set $\ell = g_2g_3$. Then $\ell \in J$ because J is a right ideal, and $\ell(x) = g_2(u) = y$. On the other hand $R(\ell) \subseteq R(g_2)$ so $v \notin R(\ell)$ and therefore $\ell h^{-1}(x) \neq fh^{-1}(x)$.

Using (i), (ii) and the definition of ϕ we get $\phi(f)(x) = ef(x) = e(x) = y$ whence $hfh^{-1}(x) = y$ and

$$\text{(iii)} \quad fh^{-1}(x) = h^{-1}(y).$$

Similarly $\phi(\ell)(x) = \ell(x) = y$ whence $h\ell h^{-1}(x) = y$ and

$$\text{(iv)} \quad \ell h^{-1}(x) = h^{-1}(y).$$

But now (iii) and (iv) imply $\ell h^{-1}(x) = fh^{-1}(x)$, contradicting (ii). We conclude ϕ is an outer automorphism. □

<u>REMARK</u>. The outer automorphism ϕ in Example 3.2.1 has this form for $J = I_2$, $e = (x,y)$. □

<u>THEOREM 3.2.35</u>. Let $X = W$ and $S = CT(X,\mathcal{E},p,q)$. Then all automorphisms of S are inner if and only if all $\mathcal{C}_q$ sets are λ-related, equivalently, there is only one component I_α in the right ideal decomposition of S in 3.2.19.

<u>Proof</u>. If all $\mathcal{C}_q$ sets are λ-related, then there is only one $\mathcal{M}_\alpha$, namely $\mathcal{M}_\alpha = \mathcal{C}_q$. Thus $S = I_\alpha$ and hence by Theorem 3.2.2 all automorphisms are inner.

For the converse, suppose there exist $A,B \in \mathcal{C}_q$ with A not λ-related to B. Then there is a non-trivial ρ-class U meeting A and B but A is not finitely linked to B through U. Let

$\mathcal{L}$ = collection of all $\mathcal{C}_q$ sets C for which $C \cap U \neq \Phi$
and C is finitely linked to A through U, and

$J = \{g \in S \mid R(g) \notin \mathcal{L}\}$.

Note that J is a right ideal because for $C,D \in \mathcal{C}_q$, if $C \notin \mathcal{L}$ and $D \subseteq C$ then $D \notin \mathcal{L}$. Let $A \cap U = \{x\}$. By replacing B if necessary by $(B - \{b\}) \cup \{y\}$ where $\{b\} = B \cap U$ and $y \in U$, we can assume there is $y \in B \cap U$, $y \neq x$. Let $e \in \mathcal{G}_W$ be the transposition (x,y). One easily checks that the hypotheses of Lemma 3.2.34 hold. Moreover condition (2) of the Lemma is satisfied when $R(f) = A$, $R(g) = B$. Hence the map $\phi : S \to S$ defined by

$$\phi(f) = \begin{cases} f & \text{if } f \in J \\ (x,y)f & \text{if } f \notin J \end{cases}$$

is an outer automorphism. □

COROLLARY 3.2.36. For any Croisot-Teissier semigroup $S = CT(X,\mathcal{E},p,q)$, if $|\mathcal{E}| < p$ then $X = W$ and all automorphisms of S are inner.

Proof. Let $r = |\mathcal{E}| < p$. For $C \subseteq X$ and $x \in X$, let us define the *releasing set of* x *in* C by

$$R(x,C) = \{y \in C \mid y \; \mathcal{E}_i \; x \text{ for some } i \in I\} .$$

Observe that when C is well-separated, $|R(x,C)| \leqslant |\mathcal{E}| = r < p$ so that $(C - R(x,C)) \cup \{x\}$ is a well-separated set containing x. Thus $X = W$. By Theorem 3.2.35, to show automorphisms of S are inner amounts to showing $A \;\lambda\; B$ for all C_q sets A and B. We prove a little more, namely:

(3.2.4) there exist C_q sets $A_1 \subseteq A$ and $B_1 \subseteq B$ such that $A_1 \cup (A \cap B) \cup B_1$ is a well-separated set.

Since any set of ordinal numbers is well-ordered, there exists a least element ξ amongst all ordinals of cardinality p. Let L be a well-ordered set of cardinality p, with the ordinal number of L (ord L) equal to ξ. Then for each $\ell \in L$, the *initial segment* of ℓ

$$s(\ell) = \{j \in L \mid j < \ell\}$$

has cardinality less than p. (Else, if $|s(\ell_0)| \geqslant p$, some $\ell_0 \in L$, then ord $s(\ell_0) \geqslant \xi$ because of the choice of ξ. However for the initial segment $s(\ell_0)$ we must have ord $s(\ell_0) <$ ord $L = \xi$, and so $\xi \leqslant$ ord $s(\ell_0) < \xi$, a contradiction). We construct by transfinite induction a set $Y = \{b_\ell\}_{\ell \in L}$ of p elements of B with disjoint releasing sets in A.

Choose any $b_0 \in B$ for the first element of Y. Now assume the b_j have been constructed for all $j < \ell$. Let

$$Z = \bigcup_{j<\ell} R(b_j, A).$$

Since $|R(b_j,A)| \leqslant r < p$ and $|s(\ell)| < p$, we have $|Z| < p$. Suppose $R(b,A) \cap Z \neq \Phi$ for all $b \in B$. Then $B \subseteq \bigcup_{x \in Z} R(x,B)$ and so

$$|B| \leqslant |Z|r < p, \text{ because for each } x \in Z, |R(x,B)| \leqslant r,$$

contradicting $|B| = p$. Hence there exists $b_\ell \in B$ with $R(b_\ell,A) \cap Z = \Phi$. This completes the construction of Y.

Set $Y_1 = \{y \in Y \mid R(y,A) = \Phi\}$ and consider two cases.

Case (i). $|Y_1| = p$.

Since $Y_1 \subseteq Y \subseteq B$, $Y_1 \in \mathcal{C}_q$. Let $A_1 = A$ and $B_1 = Y_1$, then $A_1 \cup (A \cap B) \cup B_1 = A \cup B_1 \in \mathcal{C}_q$, because for every $y \in B_1$, $R(y,A) = \Phi$.

Case (ii). $|Y_1| < p$.

Write $Y = Y_2 \dot{\cup} Y_3$ with $Y_1 \subseteq Y_2$ and $|Y_2| = |Y_3| = p$. Let $A_1 = \bigcup_{y \in Y_3} R(y,A)$ and $B_1 = Y_2$. Then $A_1 \subseteq A$ with $|A_1| \geqslant |Y_3| = p$ and $B_1 \subseteq Y \subseteq B$ with $|B_1| = p$, that is $A_1, B_1 \in \mathcal{C}_q$. Also for every $b \in B_1$, $R(b,A_1) = \Phi$. (Indeed, since Y consists of elements with disjoint releasing sets in A, $A_1 \cap R(b,A) = \Phi$. Now, $R(b,A_1) \subseteq R(b,A)$, for $A_1 \subseteq A$, and so

$$A_1 \cap R(b,A_1) = \Phi.$$

This in conjunction with $R(b,A_1) \subseteq A_1$ implies $R(b,A_1) = \Phi$.) We conclude that $A_1 \cup (A \cap B) \cup B_1$ is a well-separated set.

The assertion in (3.2.4) follows.

By a straightforward (and by now familiar) argument, it follows that $A \lambda B$. □

<u>EXAMPLE 3.2.37.</u> We give a variation on Example 3.2.1 which provides a non-reduced example with a single maximal λ-related family. Theorem 3.2.35 then assures us that all automorphisms are inner.

Recall that $X = C \cup D$ where $|C| = |D| = p$ and $C \cap D = \{x,y\}$. Select $\{v,w\} = V \subseteq C - D$. For $c \in C - D$, $d \in D - C$ let $\&_{c,d} = \{\{c,d\}, V, \text{singletons}\}$ when $c \in C - V$, and $\&_{v,d} = \&_{w,d} = \{\{d\} \cup V, \text{singletons}\}$. Let $\& = \{\&_{c,d}\}$ and $S = CT(X,\&,p,q)$. Since V is a non-trivial ρ-class, S is not reduced. Well-separated sets are those subsets of cardinality p either in D, or in C and including at most one element of V. Hence $W = X$. If $A,B \in \mathcal{C}_q$ and meet V, then $A,B \subseteq C$ so are finitely linked through V. Thus there is a single component in the right ideal decomposition. Notice however that not all $\mathcal{C}_q$ sets A and B are finitely linked through any common *singleton* ρ-class. □

For later sections we need to relate automorphisms of $CT(W,\&,p,q)$ to those of $CT(W,\&,p,p)$. We shall establish a natural correspondence (Proposition 3.2.40). To do this we first relate maximal families of λ-related $\mathcal{C}_q$ sets and the A_α decomposition of W in Proposition 3.2.19, in the general case $p \geqslant q$, to the case $p = q$.

<u>LEMMA 3.2.38.</u> The correspondence

$$M \mapsto M \cap \mathcal{C}_p$$

is a 1-1 correspondence between maximal families of λ-related $\mathcal{C}_q$ sets and maximal families of λ-related $\mathcal{C}_p$ sets.

<u>Proof.</u> It is easy to see that restricting the λ-relation on $\mathcal{C}_q$ to $\mathcal{C}_p$ does indeed give the λ-relation on $\mathcal{C}_p$ (because finite $\mathcal{C}_q$ links can always be replaced by suitable $\mathcal{C}_p$ links). Thus we can use the same notation for each.

Let M be a maximal family of λ-related $\mathcal{C}_q$ sets. Clearly $M \cap \mathcal{C}_p$ consists of λ-related $\mathcal{C}_p$ sets. Suppose $(M \cap \mathcal{C}_p) \cup \{B\}$ is also a λ-related family for some $B \in \mathcal{C}_p$.

Claim: $B \lambda C$ *for all* $C \in M$.

For if U is a non-trivial ρ-class meeting B and C, choose $C_1 \in \mathcal{C}_p$, $C_1 \subseteq C$ such that U meets C_1. Then $C_1 \in M \cap \mathcal{C}_p$ by Proposition 3.2.17(2), so $B \lambda C_1$ implies B and C are finitely linked through U. Thus $B \lambda C$.

Now by maximality of M, $B \in M \cap \mathcal{C}_p$. This shows $M \cap \mathcal{C}_p$ is a maximal family of λ-related $\mathcal{C}_p$ sets.

A similar argument establishes that if N is a maximal family of λ-related $\mathcal{C}_p$ sets, then

$$N' = \{B \in \mathcal{C}_q \mid A \in N \text{ whenever } A \in \mathcal{C}_p,\ A \subseteq B\}$$

is a maximal family of λ-related $\mathcal{C}_q$ sets. Indeed, assume $N' \cup \{B\}$ is also a λ-related family for some $B \in \mathcal{C}_q$. Then a $\mathcal{C}_p$ set $B_1 \subseteq B$ is in N (by Proposition 3.2.17(2)), and so $B \in N'$. Also by 3.2.17(2) $N = N' \cap \mathcal{C}_p$. In particular, taking $N = M \cap \mathcal{C}_p$ we get $M \subseteq N'$ by 3.2.17(2), whence $M = N'$ because M is maximal. Thus the correspondence

$$N \mapsto N'$$

is an inverse of $M \mapsto M \cap \mathcal{C}_p$. □

PROPOSITION 3.2.39. If $\{M_\alpha\}_{\alpha\in\Omega}$ is the collection of all maximal families of λ-related $\mathcal{C}_q$ sets, and $N_\alpha = M_\alpha \cap \mathcal{C}_p$, then:

(1) $\{N_\alpha\}_{\alpha\in\Omega}$ is the collection of all maximal families of λ-related $\mathcal{C}_p$ sets.

(2) $\bigcup_{M_\alpha} A = A_\alpha = \bigcup_{N_\alpha} A$

(so that the components of the decomposition of W in Proposition 3.2.19, in the general case $p \geqslant q$, are the same as for the case $p = q$).

(3) $\{h_\alpha\}_{\alpha\in\Omega}$ is a compatible system of bijections of W for $CT(W,\mathcal{E},p,q)$ if and only if $\{h_\alpha\}_{\alpha\in\Omega}$ is a compatible system for $CT(W,\mathcal{E},p,p)$.

Proof. (1) This is the content of Lemma 3.2.38.

(2) This follows from the observation that each $\mathcal{C}_q$ set is a

union of C_p sets, and hence by Proposition 3.2.17(2) each M_α set is a union of N_α sets.

(3) By a very similar argument to (2), $h_{\alpha|A} = h_{\beta|A}$ $\forall A \in M_\alpha \cap N_\beta$ if and only if $h_{\alpha|A} = h_{\beta|A}$ $\forall A \in N_\alpha \cap N_\beta$. SEE ERRATA □

PROPOSITION 3.2.40. Assume W = X. Then each automorphism ϕ of CT(X,&,p,p) has a unique extension to an automorphism ϕ_1 of CT(X,&,p,q), namely, if $\{h_\alpha\}_{\alpha\in\Omega}$ is a compatible system of bijections of W associated with ϕ, then

$$\phi_1(f) = h_\alpha f h_\alpha^{-1}, \text{ for all } f \in I_\alpha .$$

All automorphisms ϕ_1 of CT(X,&,p,q) arise this way. In particular

$$\text{Aut CT}(p,p) \cong \text{Aut CT}(p,q).$$

Proof. Let S = CT(W,&,p,q). Given $\phi \in$ Aut CT(p,p), there exists by Theorem 3.2.2 a compatible system $\{h_\alpha\}_{\alpha\in\Omega}$ such that

$$\phi(f) = h_\alpha f h_\alpha^{-1} \quad \text{if} \quad R(f) \in N_\alpha .$$

Define $\phi_1 : S \to S$ by

$$\phi_1(f) = h_\alpha f h_\alpha^{-1} \quad \text{if} \quad R(f) \in M_\alpha .$$

By Proposition 3.2.39(3) $\{h_\alpha\}_{\alpha\in\Omega}$ is a compatible system for S, so that the converse of Theorem 3.2.2 implies that $\phi_1 \in$ Aut S. Since $N_\alpha \subseteq M_\alpha$, ϕ_1 extends ϕ.

To show ϕ_1 is unique, suppose $\phi_2 \in$ Aut S is any extension of ϕ. Let $f \in I_\alpha$ (that is, $R(f) \in M_\alpha$). Let $x \in W$ and write $x = \phi(\ell)(v)$ for some $\ell \in$ CT(p,p) and $v \in W$. This is possible because CT(p,p) acts transitively on W. Suppose $R(\ell) \in N_\beta$. Now

$$\begin{aligned} \phi_2(f)(x) &= \phi_2(f)\phi(\ell)(v) \\ &= \phi_2(f)\phi_2(\ell)(v) && \text{because } \phi_2 \text{ extends } \phi \\ &= \phi_2(f\ell)(v) \\ &= \phi(f\ell)(v) && \text{because } f\ell \in \text{CT}(p,p) \end{aligned}$$

$= h_\alpha(f\ell)h_\alpha^{-1}(v)$ because $R(f\ell) \in M_\alpha \cap C_p = N_\alpha$

$= (h_\alpha f h_\alpha^{-1})(h_\alpha \ell h_\alpha^{-1})(v)$

$= (h_\alpha f h_\alpha^{-1})(h_\beta \ell h_\beta^{-1})(v)$ by Remark 3.2.26

$= (h_\alpha f h_\alpha^{-1})\phi(\ell)(v)$ because $R(\ell) \in N_\beta$

$= \phi_1(f)(x)$ because $R(f) \in M_\alpha$.

Hence $\phi_2 = \phi_1$.

Clearly the map $\phi \mapsto \phi_1$ is a group monomorphism of Aut CT(p,p) into Aut CT(p,q). For any $\phi_1 \in$ Aut S, we have $\phi_1(CT(p,p)) = CT(p,p)$ because CT(p,p) is the minimal ideal of S. In consequence ϕ_1 induces an automorphism of CT(p,p) and the above map is therefore an isomorphism. □

REMARK 3.2.41.

(1) Of course as a corollary of Proposition 3.2.40, when W = X

$$\text{Aut } CT(p,q) \cong \text{Aut } CT(p,r)$$

for *any* infinite cardinals $q \leqslant r \leqslant p$.

(2) When $X \neq W$, in general

$$\text{Aut } CT(p,q) \ncong \text{Aut } CT(p,p).$$

Automorphisms can be extended but not uniquely. This is shown in Section 3.5. □

3.3 AUTOMORPHISMS WHEN W PROVIDES $\mathcal{E}$ CROSS-SECTIONS

Throughout this third section we consider Croisot-Teissier semigroups $S = CT(X,\mathcal{E},p,q)$ for which W *provides* $\mathcal{E}$ *cross-sections*. That is, W meets each equivalence class of each equivalence relation: for all $i \in I$ and $C \in X/\mathcal{E}_i$, $C \cap W \neq \Phi$. Note that semigroups for which $W = X$, dealt with in Section 3.2, form a special case. Our aim is to associate with S a factor semigroup Q (again a Croisot-Teissier semigroup) for which $W = X$, and show that an automorphism of S can be described using an automorphism of Q and a related permutation z of $\mathcal{E}$. We pause to give an example in which W provides $\mathcal{E}$ cross-sections, yet $W \neq X$.

EXAMPLE 3.3.1. Let $X = C \cup D$ where $|C| = p$, $D \neq \Phi$ and $C \cap D = \Phi$. For each $c \in C$ define a partition $\mathcal{E}_c = \{\{c\} \cup D, \text{singletons}\}$ on X and let $\mathcal{E} = \{\mathcal{E}_c | c \in C\}$. Then well-separated sets for $CT(X,\mathcal{E},p,q)$ are precisely the subsets of C of cardinality p. So $W = C \neq X$ while W does provide $\mathcal{E}$ cross-sections. □

The following self-evident statement will be a key tool in the sequel.

REMARK 3.3.2. Let S have the property that W provides $\mathcal{E}$ cross-sections, and take $f,g \in S$. Then

$$f = g \text{ if and only if } \pi(f) = \pi(g) \text{ and } f|_W = g|_W. \qquad \square$$

Two items must be introduced before a statement of the main theorem for this section can be made: the Croisot-Teissier semigroup Q (for which $W = X$) associated with S, and the permutation $\tilde{\phi}$ of $\mathcal{E}$ associated with an automorphism ϕ of S.

For the first, define $\mathcal{E}_i' = \mathcal{E}_i \cap (W \times W)$ if thinking in terms of equivalences, or $\mathcal{E}_i' = \{C \cap W \mid C \in \mathcal{E}_i\}$ if thinking in terms of partitions, and let $\mathcal{E}' = \{\mathcal{E}_i'\}_{i \in I}$. Note that $|W| \geqslant p$, $|W/\mathcal{E}_i'| = p$ for each $i \in I$, and that well-separated sets for W and $\mathcal{E}'$ are identical to those for X and $\mathcal{E}$. These facts ensure that $Q = CT(W,\mathcal{E}',p,q)$ is a non-empty Croisot-Teissier semigroup. In Section 3.2 the reduced semigroup $\bar{\bar{S}}$ (handled in Section 3.1) occurred as a factor semigroup of S, and was instrumental in yielding the structure of automorphism of S. Dually here we show that Q (handled in Section 3.2) is a factor semigroup of S, and relate automorphisms of S to those of Q. For the second item, we will show in Proposition 3.3.8 that every $\phi \in \text{Aut } S$ induces a permutation $\tilde{\phi}$ of $\mathcal{E}$ via $\tilde{\phi}(A) = \pi(\phi(f))$, where $f \in S$ and $\pi(f) = A$.

Finally, we say that $z \in G_{\mathcal{E}}$ is a *lifting* of $z' \in G_{\mathcal{E}'}$ if $\gamma z = z'\gamma$, where $\gamma : \mathcal{E} \to \mathcal{E}'$ is given by $\gamma(\mathcal{E}_i) = \mathcal{E}'_i$, for each $i \in I$. That is, the diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "z"] \arrow[d, "\gamma"'] & \mathcal{E} \arrow[d, "\gamma"'] \\
\mathcal{E}' \arrow[r, "z'"] & \mathcal{E}'
\end{tikzcd}$$

commutes.

<u>THEOREM 3.3.3</u>. Let $S = CT(X,\mathcal{E},p,q)$ be a Croisot-Teissier semigroup for which W provides $\mathcal{E}$ cross-sections, and let ϕ be an automorphism of S. Then there exist, uniquely,

(1) an $\psi \in \text{Aut } CT(W,\mathcal{E}',p,q)$, and

(2) a $z \in G_{\mathcal{E}}$ lifting $\tilde{\psi}$

such that

(i) $\phi(f)|_W = \psi(f|_W)$, and

(ii) $\pi(\phi(f)) = z(\pi(f))$, for all $f \in S$.

Conversely, given S and (1) and (2), there exists a unique automorphism ϕ of S such that (i) and (ii) hold. □

We prepare for the proof of this theorem.

Define $\theta : S \to Q$ via $\theta(f) = f|_W$. Since $f|_W$ is a selfmap of W $\pi(f|_W) = \pi(f)' \in \mathcal{E}'$ and $R(f|_W) = R(f) \in C_q$, certainly $f|_W \in Q$.

LEMMA 3.3.4. (1) θ is a homomorphism from S onto Q.

(2) $\ker\theta = \{(f,g) \mid f\ell = g\ell \text{ for all } \ell \in S\}$.

Proof. (1) For $f,g \in S$ we have $fg|_W = f|_W g|_W$ from which it follows that θ is a homomorphism. Given $k \in Q$, take $A \in \&$ with $A' = \pi(k)$. Then there is a unique $f \in S$ (see Remark 3.3.2) such that $\pi(f) = A$ and $f|_W = k$, so θ is onto.

(2) We have $(f,g) \in \ker\theta$ iff $f|_W = g|_W$. If $f|_W = g|_W$, immediately $f\ell = g\ell$ for all $\ell \in S$, since $R(\ell) \subseteq W$. For the converse, choose $w \in W$ and $\ell \in S$ with $w \in R(\ell)$. Then $f\ell = g\ell$ implies $f(w) = g(w)$, whence $f|_W = g|_W$. □

By the remark prior to Definition 3.2.8, $\ker\theta$ is a congruence on S.

DEFINITION 3.3.5. Let ν be the congruence on S given by

$$f \nu g \text{ if and only if } f\ell = g\ell, \text{ for all } \ell \in S.$$ □

From Lemma 3.3.4 we at once have

PROPOSITION 3.3.6. $$S/\nu \cong Q.$$ □

LEMMA 3.3.7. (1) Let $f,g \in CT(X,\&,p,p) \subseteq S$. Then, $\pi(f) = \pi(g)$ if and only if $f = kg$ for some $k \in S$.

(2) Let $f,g \in S$. Then, $\pi(f) = \pi(g)$ if and only if $rf = sg$ for some $r,s \in S$.

Proof. (1) If $f = kg$, then $\pi(f) = \pi(kg) = \pi(g)$. Conversely, assume $\pi(f) = \pi(g)$ and choose $A \in \&$ and $Y \subseteq X$ such that $Y \dot{\cup} R(g)$ is a cross-section of A. Note that $|Y| = p$, since $g \in CT(X,\&,p,p)$. Choose $A \in \mathcal{C}_p$ with $R(f) \subseteq A$ and $|A - R(f)| = p$, and let ℓ be any bijection from Y onto $A - R(f)$. Define the function k on X as follows: for $x \in X$,

$$k(x) = \begin{cases} \ell(y), & \text{if} \quad x\ A\ y \text{ for some } y \in Y, \\ f(z), & \text{if} \quad x\ A\ g(z). \end{cases}$$

That $\pi(f) = \pi(g)$ ensures k is well-defined, $\pi(k) = A$ and $R(k) = A$, so $k \in S$ and certainly $kg = f$.

This proof is presented for the sake of completeness. By drawing on results in [3,4] an alternative proof of this converse is available: suppose $\pi(f) = \pi(g) = \mathcal{E}_i$, so $f,g \in T_i \cap CT(X,\mathcal{E},p,p)$, a minimal left ideal of $CT(X,\mathcal{E},p,p)$, [4,p.87], and hence a left simple semigroup. Thus [3,p.6] there exists $k \in T_i \cap CT(X,\mathcal{E},p,p)$ such that $f = kg$.

(2) If $rf = sg$, then $\pi(f) = \pi(rf) = \pi(sg) = \pi(g)$. Conversely, choose any $r,t \in CT(X,\mathcal{E},p,p)$, so $rf,tg \in CT(X,\mathcal{E},p,p)$. Now $\pi(rf) = \pi(tg)$ so by (1) there exists $k \in S$ such that $rf = k(tg)$. Take $s = kt$ to complete the proof. □

<u>PROPOSITION 3.3.8.</u> Let $\phi \in \text{Aut } S$. Then $\tilde{\phi} : \mathcal{E} \to \mathcal{E}$ given by

$$\tilde{\phi}(A) = \pi(\phi(f)), \text{ for any } f \in S \text{ with } \pi(f) = A\ ,$$

is a well-defined bijection of $\mathcal{E}$.

<u>Proof.</u> It follows at once from Lemma 3.3.7 (2) that for $f,g \in S$, $\pi(f) = \pi(g)$ if and only if $\pi(\phi(f)) = \pi(\phi(g))$, whence $\tilde{\phi}$ is well-defined. Similarly, the automorphism ϕ^{-1} induces a map $\widetilde{\phi^{-1}}$ of $\mathcal{E}$ via $\widetilde{\phi^{-1}}(A) = \pi(\phi^{-1}(f))$, for any $f \in S$ with $\pi(f) = A$. Then for an $A \in \mathcal{E}$,

$$\begin{aligned} \widetilde{\phi^{-1}}\tilde{\phi}(A) &= \widetilde{\phi^{-1}}(\pi(\phi(f))), \text{ for } f \in S \text{ with } \pi(f) = A \\ &= \pi(\phi^{-1}\phi(f)) \\ &= \pi(f) = A\ . \end{aligned}$$

Since also $\tilde{\phi}\widetilde{\phi^{-1}}(A) = A$, each $A \in \mathcal{E}$, $\widetilde{\phi^{-1}}$ is the inverse of $\tilde{\phi}$ and so $\tilde{\phi}$ is a bijection of $\mathcal{E}$. □

<u>REMARK 3.3.9.</u> Both 3.3.7 and 3.3.8 hold generally: no use in their proof was made of the assumption that W provides $\mathcal{E}$ cross-sections. □

Proof of Theorem 3.3.3. Let $\phi \in \text{Aut } S$. From 3.3.6, $S/\nu \cong Q$ and since $f\ell = g\ell$ for all $\ell \in S$ iff $\phi(f)\ell = \phi(g)\ell$ for all $\ell \in S$, ϕ permutes classes of ν. Thus ϕ on S induces an automorphism ψ on Q such that

$$\psi(\theta(f)) = \theta(\phi(f)) \quad \text{or} \quad \psi(f|_W) = \phi(f)|_W, \text{ for all } f \in S,$$

establishing (i).

Choose $z = \tilde{\phi}$ as in 3.3.8. We show that $\tilde{\phi}$ lifts $\tilde{\psi}$. Let $A \in \mathcal{E}$ and choose $f \in S$ with $\pi(f) = A$. Note that $\pi(f|_W) = \gamma(\pi(f))$. Then

$$\begin{aligned} \tilde{\psi}\gamma(A) &= \tilde{\psi}(\pi(f|_W)) \\ &= \pi(\psi(f|_W)), && \text{from the definition of } \tilde{\psi} \\ &= \pi(\phi(f)|_W), && \text{from the definition of } \psi \\ &= \gamma(\pi(\phi(f))), && \text{using the above note again} \\ &= \gamma(\tilde{\phi}(\pi(f))), && \text{from the definition of } \tilde{\phi} \\ &= \gamma\tilde{\phi}(A). \end{aligned}$$

Now (ii) holds, by the definition of $\tilde{\phi}$. Finally, conditions (i) and (ii) of the theorem statement ensure ψ and z are unique.

For the converse, take $\psi \in \text{Aut } Q$ and $z \in G_{\mathcal{E}}$, a lifting of $\tilde{\psi}$. With $f \in S$,

$$\begin{aligned} \gamma z(\pi(f)) &= \tilde{\psi}\gamma(\pi(f)), && \text{since } z \text{ is a lifting of } \tilde{\psi} \\ &= \tilde{\psi}\pi(f|_W), && \text{from the definition of } \gamma \\ &= \pi(\psi(f|_W)), && \text{from the definition of } \tilde{\psi}. \end{aligned}$$

This guarantees the consistency of the following construction: let $\phi(f)$ be the selfmap of X with $\pi(\phi(f)) = z(\pi(f))$ and $\phi(f)|_W = \psi(f|_W)$. Since $\pi(\phi(f)) \in \mathcal{E}$ and $R(\phi(f)) = R(\psi(f|_W)) \in C_q$, $\phi(f) \in S$, while Remark 3.3.2 confirms that $\phi(f)$ is uniquely determined. We complete the proof of the theorem by showing that ϕ, the well-defined mapping on S taking f to this $\phi(f)$, is an automorphism.

Take $f,g \in S$. Then

$$\pi(\phi(f)\phi(g)) = \pi(\phi(g)) = z(\pi(g)) = z(\pi(fg)) = \pi(\phi(fg)), \text{ while}$$

$$\begin{aligned}\phi(f)\phi(g)|_W &= \phi(f)|_W\phi(g)|_W = \psi(f|_W)\psi(g|_W)\\ &= \psi((f|_W)(g|_W)) = \psi((fg)|_W)\\ &= \phi(fg)|_W .\end{aligned}$$

By appealing again to 3.3.2 we have that $\phi(f)\phi(g) = \phi(fg)$ for all $f,g \in S$.

Since z lifts $\tilde{\psi}$, $\gamma z = \tilde{\psi}\gamma$, whence $(\tilde{\psi})^{-1}\gamma = \gamma z^{-1}$. But $(\tilde{\psi})^{-1} = \widetilde{\psi^{-1}}$, so z^{-1} lifts $\widetilde{\psi^{-1}}$. Repeating the previous argument with ψ^{-1} and z^{-1} we can produce a homomorphism χ of S such that

$$\pi(\chi(f)) = z^{-1}(\pi(f)) \quad\text{and}\quad \chi(f)|_W = \psi^{-1}(f|_W) \text{ for all } f \text{ in } S.$$

Now
$$\pi(\chi\phi(f)) = z^{-1}(z(\pi(f))) = \pi(f), \text{ and}$$
$$\chi\phi(f)|_W = \psi^{-1}(\psi(f|_W)) = f|_W$$

so a final application of 3.3.2 gives that $\chi\phi$ is the identity on S. Similarly $\phi\chi$ is the identity, and ϕ is invertible. □

<u>REMARK</u>. We construct an example of $\tilde{\psi}$ for which no lifting z exists (where $\psi \in \text{Aut } Q$). It follows that the natural homomorphism $\phi \mapsto \psi$ from Aut S to Aut Q is not onto. Consider Example 3.3.1 in which $X = C \dot{\cup} D$, $|C| = p$, $D = \{d_1, d_2\}$ and for every $c \in C$, $\mathcal{E}_c = \{\{c\} \cup D, \text{singletons}\}$. Fix x,y,u,v in C and let $\mathcal{E}_1 = \{\{x,y,d_1\}, \{u,d_2\}, \text{singletons}\}$, $\mathcal{E}_2 = \{\{x,y\} \cup D, \text{singletons}\}$ and $\mathcal{E}_3 = \{\{u,v\} \cup D, \text{singletons}\}$. Let $\mathcal{E} = \{\mathcal{E}_c \mid c \in C\} \cup \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$. Here $W = C$ and an $h \in G_W$ is p.p. iff

$$\begin{aligned}&\text{either} && h(\{x,y\}) = \{x,y\},\ h(\{u,v\}) = \{u,v\}\\ &\text{or} && h(\{x,y\}) = \{u,v\},\ h(\{u,v\}) = \{x,y\}.\end{aligned}$$

Choose an h of the latter type, then by 3.1.6 ψ_h is an automorphism of Q. Now, $\gamma(\mathcal{E}_1) = \gamma(\mathcal{E}_2)$, and because of the choice of h we have that

$$\tilde{\psi}_h\gamma(\mathcal{E}_1) = \tilde{\psi}_h\gamma(\mathcal{E}_2) = \gamma(\mathcal{E}_3).$$

Assume there exists $z \in G_{\mathcal{E}}$ such that $\gamma z = \tilde{\psi}_h\gamma$. Then

$$\gamma z(\mathcal{E}_1) = \gamma z(\mathcal{E}_2) = \gamma(\mathcal{E}_3).$$

Since there is no $A \in \mathcal{E} - \{\mathcal{E}_3\}$ with $\gamma(A) = \gamma(\mathcal{E}_3)$, we conclude that $z(\mathcal{E}_1) = z(\mathcal{E}_2)$, or $z \notin G_{\mathcal{E}}$. □

In order to illustrate Theorem 3.3.3 we build on the early example of this section.

EXAMPLE 3.3.10. Recall Example 3.3.1 where we introduced $S = CT(X,\mathcal{E},p,q)$ with $X = C \dot{\cup} D$, $|C| = p$, $D \neq \Phi$ and $\mathcal{E} = \{\mathcal{E}_c \mid c \in C\}$ such that $W = C$ and W provides $\mathcal{E}$ cross-sections. Here $\mathcal{E}_c \cap (W \times W) = \iota_W$ for each $c \in C$, so $\mathcal{E}' = \{\iota_W\}$ and $Q = CT(W,\{\iota_W\},p,q)$. Thus Q is reduced and a union of Baer-Levi semigroups, as detailed in Example 3.1.3. Since Q is reduced, all automorphisms of Q are inner (Theorem 3.1.2). Moreover, because every element of G_W is p.p., Aut $Q \cong G_W$ by 3.1.6.

Utilizing Theorem 3.3.3 we have that each automorphism ϕ of S can be associated with an (h,z) pair where $h \in G_W$ and $z \in G_{\mathcal{E}}$, such that

(3.3.1) $\quad \phi(f)|_W = hfh^{-1}$ and $\pi(\phi(f)) = z(\pi(f))$, for all $f \in S$.

Thus we define a map

$$\theta : \text{Aut } S \to G_W \times G_{\mathcal{E}},$$

given by $\theta(\phi) = (h,z)$, where $h \in G_W$ and $z \in G_{\mathcal{E}}$ satisfy (3.3.1).

Conversely, any pair h and z with $h \in G_W$, $z \in G_{\mathcal{E}}$ (and hence any element of $G_W \times G_{\mathcal{E}}$) determines a $\phi \in$ Aut S such that (3.3.1) holds because, with $\mathcal{E}'$ a singleton, each bijection z of $\mathcal{E}$ is a lifting of $\tilde{\psi}$, $\psi \in$ Aut Q. This allows us to define a map

$$\chi : G_W \times G_{\mathcal{E}} \to \text{Aut } S$$

given by $\chi((h,z)) = \phi$, where $h \in G_W$ and $z \in G_{\mathcal{E}}$ determine $\phi \in$ Aut S via (3.3.1). Certainly χ is the inverse of θ, and so θ is a bijection. We show it is an isomorphism. Let $\phi_1, \phi_2 \in$ Aut S and

$$\theta(\phi_1) = (h_1,z_1), \ \theta(\phi_2) = (h_2,z_2).$$

Then $\theta(\phi_1)\theta(\phi_2) = (h_1h_2, z_1z_2)$. Thus $\theta(\phi_1)\theta(\phi_2)$ determines an automorphism ϕ with

$$\phi(f)\big|_W = h_1h_2f(h_1h_2)^{-1} = \phi_1\phi_2(f)\big|_W \quad \text{and}$$
$$\pi(\phi(f)) = z_1z_2(\pi(f)) = z_1(\pi(\phi_1(f))) = \pi(\phi_1\phi_2(f)),$$

for all f in S. But the same automorphism is determined by $\theta(\phi_1\phi_2)$. We conclude that $\theta(\phi_1\phi_2) = \theta(\phi_1)\theta(\phi_2)$, and hence

$$\mathrm{Aut}\ S \cong G_W \times G_{\mathcal{E}}$$

for this example. □

Despite the special nature of this example, outer automorphisms exist when $|D| < \infty$, as the corollary to our next proposition reveals. The proposition itself continues the theme of Theorem 3.2.35. Let k be partition-preserving relative to $\mathcal{E}$. The associated bijection $\tilde{k}$ of $\mathcal{E}$ is given by $\tilde{k}(A) = \{k(C) \mid C \in A\}$, for $A \in \mathcal{E}$, (here viewing A as a partition).

DEFINITION 3.3.11. The semigroup $S = CT(X,\mathcal{E},p,q)$ has the *extension property* if for each p.p. $h \in G_W$ (relative to $\mathcal{E}'$) and lifting $z \in G_{\mathcal{E}}$ of $\tilde{h}$, there exists a p.p. $k \in G_X$ extending h such that $z = \tilde{k}$. □

PROPOSITION 3.3.12. Let $S = CT(X,\mathcal{E},p,q)$ be such that W provides $\mathcal{E}$ cross-sections. Then Aut S = Inn S if and only if

Aut Q = Inn Q and S has the extension property.

(Recall from Theorem 3.2.35 that Aut Q = Inn Q iff all C_q sets are λ-related.)

Proof. Assume Aut S = Inn S. We begin by showing that S has the extension property. Take h, p.p. in G_W, and a lifting $z \in G_{\mathcal{E}}$. Consider ψ, the inner automorphism of Q determined by h. Then Theorem 3.3.3 ensures $\phi \in$ Aut S exists such that

$$\pi(\phi(f)) = z(\pi(f)) \quad \text{and} \quad \phi(f)|_W = hfh^{-1}, \text{ for all } f \in S.$$

Since ϕ is inner, we can suppose p.p. $k \in G_X$ induces ϕ. Then $z = \tilde{\phi} = \tilde{k}$ lifts $\tilde{h}$, while $k|_W \in G_W$ since k and k^{-1} leave W invariant. Let $f \in Q$. Choose an extension $f_1 \in S$ of f. Then

$$hfh^{-1} = \phi(f_1)|_W = (kf_1k^{-1})|_W = (k|_W)f(k|_W)^{-1}.$$

By 3.1.6, $k|_W = h$ so k extends h. Thus S has the extension property.

Now suppose Aut $Q \neq$ Inn Q. Then as in the proof of Theorem 3.2.35 there will exist an outer automorphism ψ of Q with $\tilde{\psi} = \iota_{\mathcal{E}'}$. Certainly $z = \iota_{\mathcal{E}}$ lifts $\iota_{\mathcal{E}'}$ so take $\phi \in$ Aut S associated with ψ and $\iota_{\mathcal{E}}$, as in Theorem 3.3.3. Since ϕ is inner there exists a p.p. $k \in G_X$ such that $\phi(f) = kfk^{-1}$ for all $f \in S$. Let $h = k|_W \in G_W$. Take $g \in Q$ and an extension $f \in S$. Then

$$\begin{aligned}\psi(g) &= \psi(f|_W)\\ &= \phi(f)|_W\\ &= kfk^{-1}|_W\\ &= hfh^{-1}\\ &= hgh^{-1},\end{aligned}$$

so ψ is inner, a contradiction. Thus Aut Q = Inn Q.

For the converse, take $\phi \in$ Aut S, and suppose it is associated with $\psi \in$ Aut Q and $z \in G_{\mathcal{E}}$. Then $\psi(g) = hgh^{-1}$ for all $g \in Q$ and some p.p. $h \in G_W$. Now $\tilde{\psi} = \tilde{h}$ so z lifts $\tilde{h}$, whence from the extension property there exists a p.p. $k \in G_X$ such that $z = \tilde{k}$ and $h = k|_W$. Also,

$$\pi(\phi(f)) = z(\pi(f)) = \tilde{k}(\pi(f)) = \pi(kfk^{-1}), \quad \text{and}$$
$$\phi(f)|_W = \psi(f|_W) = hfh^{-1} = (kfk^{-1})|_W$$

for each $f \in S$. Hence $\phi(f) = kfk^{-1}$, using 3.3.2. □

A contrast with Corollary 3.2.36 is provided by the following consequence of Proposition 3.3.12.

COROLLARY 3.3.13. Let W provide $\mathcal{E}$ cross-sections for $S = CT(X,\mathcal{E},p,q)$. If $|\mathcal{E}'| < p$ and $1 \leqslant |X - W| < \infty$, then outer automorphisms of S exist.

Proof. Early in the proof of Corollary 3.2.36 it was shown that if $|\mathcal{E}| < p$, then $X = W$, so we must here have that $|\mathcal{E}| \geqslant p$. Thus $\gamma^{-1}(A)$ is infinite for some $A \in \mathcal{E}'$, whence there exist infinitely many liftings $z \in G_{\mathcal{E}}$ of ι_W. But since $|X - W| < \infty$, only finitely many of these have the form $\tilde{k}$ for some $k \in G_X$ with $k|_W = \iota_W$. By 3.3.12, S must have outer automorphisms. □

We conclude with a simple concrete example of such an outer automorphism ϕ, for which the associated ψ is inner.

EXAMPLE 3.3.14. We revisit Example 3.3.10, assuming D a singleton, $\{d\}$ say, so $X - W = \{d\}$. By Corollary 3.3.13 we know there are outer automorphisms. Let $h = \iota_W$ and $z \in G_{\mathcal{E}}$ be the transposition $(\mathcal{E}_a, \mathcal{E}_b)$ for fixed $a,b \in C = W$. Certainly z is a lifting of $\tilde{\iota}_W$ but $z \neq \tilde{k}$ for any p.p. extension k of ι_W since ι_X is the only such map. As the proof of 3.3.12 indicates, an outer automorphism ϕ of S is provided by

$$\pi(\phi(f)) = z(\pi(f)) \quad \text{and} \quad \phi(f)|_W = f|_W, \text{ for all } f \in S.$$

That is, ϕ differs from the identity automorphism only in that it transposes the equivalences $\mathcal{E}_a$ and $\mathcal{E}_b$.

Finally, recall that if S is reduced and $X = W$, then all automorphisms of S are inner (Theorem 3.1.2). Here we have exhibited an outer automorphism for an example with S reduced and where X exceeds W by only a singleton. □

3.4. RANGE-PRESERVING AUTOMORPHISMS

In all cases considered previously all automorphisms possessed the property which we now isolate for consideration (see 3.2.2 and 3.3.3).

DEFINITION 3.4.1. An automorphism ϕ of $S = CT(X,\mathcal{E},p,q)$ is termed *range-preserving* if, for all f,g in S,

$$R(f) \subseteq R(g) \text{ if and only if } R(\phi(f)) \subseteq R(\phi(g)). \qquad \square$$

Whenever ϕ has this property we at once have the order-automorphism H associated with ϕ, given by $H(R(f)) = R(\phi(f))$, for all f in S. In the next section necessary and sufficient conditions will be presented for a Croisot-Teissier semigroup to have all its automorphisms range-preserving (Theorem 3.5.14), revealing that in general there exist automorphisms which fail to preserve ranges. From this we know that a general algebraic characterisation of the range inclusion relation cannot be found for all Croisot-Teissier semigroups.

Throughout this section we examine the structure of range-preserving automorphisms. Three components are involved in the description (Theorem 3.4.4), one additional to those required in the previous section. There, in assuming that W provided $\mathcal{E}$ cross-sections, we ensured that no classes comprising solely elements outside W could exist in any partition. Our next example shows that this need not be the case in general.

EXAMPLE 3.4.2. Let $X = C \,\dot{\cup}\, D$ with $|C| = p$ and $D \neq \Phi$. Let $\mathcal{E}_c = \{D \cup \{c\}, \text{singletons}\}$, for each $c \in C$, $\mathcal{E}_0 = \{D, \text{singletons}\}$ and $\mathcal{E} = \{\mathcal{E}_c \mid c \in C\} \cup \{\mathcal{E}_0\}$. Then $W = C$ and $\mathcal{E}_0$ has a class $D \subseteq X - W$, so W does not provide $\mathcal{E}$ cross-sections. Note that by partitioning D and changing $\mathcal{E}_0$ to the singletons on C together with the partition on D we could ensure $\mathcal{E}_0$ has up to p classes in $X - W$. $\square$

DEFINITION 3.4.3. Let $S = CT(X,\mathcal{E},p,q)$. For each $\mathcal{E}_i$, $i \in I$, define

$$B(\mathcal{E}_i) = \{E \in \mathcal{E}_i \mid E \subseteq X - W\},$$

informally termed the set of "bad" classes of $\mathcal{E}_i$. Those remaining classes, meeting W, are termed "good" classes.

Let $J = \{i \in I \mid B(\mathcal{E}_i) \neq \Phi\}$, the indices of equivalences for which W does not provide a cross-section. □

As in the previous section we will relate automorphisms on a general Croisot-Teissier semigroup $S = CT(X,\mathcal{E},p,q)$ to those on an associated semigroup where $X = W$. We briefly recall the construction of this semigroup in this unrestricted context. Let $\mathcal{E}_i' = \mathcal{E}_i \cap (W \times W)$ and $\mathcal{E}' = \{\mathcal{E}_i'\}$ be the family of derived equivalences. Again $|W| \geq p$, $|W/\mathcal{E}_i'| = p$ for each $i \in I$, and well-separated sets for W relative to $\mathcal{E}'$ are identical to those for X relative to $\mathcal{E}$. Thus $Q = CT(W,\mathcal{E}',p,q)$ is a non-empty Croisot-Teissier semigroup.

Recall that the expression of S as a union of right ideals, $S = \bigcup_{\alpha\in\Omega} I_\alpha$, remains possible in general (Proposition 3.2.19). We are now in a position to state the main result of this section.

THEOREM 3.4.4. Let $S = CT(X,\mathcal{E},p,q)$ be an arbitrary Croisot-Teissier semigroup, and let ϕ be a range-preserving automorphism of S. Then there exist, uniquely,

(1) an automorphism ψ of $CT(W,\mathcal{E}',p,q)$,

(2) a bijection $z : \mathcal{E} \to \mathcal{E}$ lifting $\tilde{\psi}$, and

(3) a family of bijections $\{y_i\}_{i\in J}$ where $y_i : B(\mathcal{E}_i) \to B(z(\mathcal{E}_i))$,

such that for any compatible system of bijections $\{h_\alpha\}_{\alpha\in\Omega}$ of W associated with ψ,

(i) $\phi(f)|_W = \psi(f|_W) = h_\alpha f h_\alpha^{-1}$, for all $f \in I_\alpha$,

(ii) $\pi(\phi(f)) = z(\pi(f))$, and

(iii) $\phi(f)(D) = h_\alpha f y_i^{-1}(D)$, for all $f \in T_i \cap I_\alpha$ and $D \in B(z(\mathcal{E}_i))$.

Conversely, given S and (1), (2) and (3), there exists a unique range-preserving $\phi \in \mathrm{Aut}\ S$ such that (i), (ii) and (iii) hold. □

Informally, the theorem shows that the partition of $\phi(f)$ is determined by z from the partition of f, and that on W (and hence all good classes) $\phi(f)$ is conjugation of f ($\in I_\alpha$) by h_α. On bad classes the h_α maps retain their range re-ordering role, while the y_i maps take over domain rearrangement formerly performed by the h_α.

In proving the theorem we follow lines similar to those of the previous section, with two exceptions. Firstly, functions in Q do not now necessarily extend to functions in S (c.f. proof of 3.3.4(1)), because of the possible presence of too many "bad" classes. Secondly, the action of $\phi(f)$ on such classes must be explained. This latter requirement gives rise to the extra component of the y_i maps in the main theorem.

<u>LEMMA 3.4.5</u>. Let ϕ be an automorphism of S. Then there exists a unique automorphism ψ of $Q = CT(W,\mathcal{E}',p,q)$ such that

$$\phi(f)|_W = \psi(f|_W), \text{ for all } f \in S .$$

<u>Proof</u>. Note that the C_r sets, $q \leqslant r \leqslant p$, of Q are precisely those of S, since the well-separated sets are common. We begin by linking ϕ with an ψ in Aut $CT(W,\mathcal{E}',p,p)$. Since $|B(\mathcal{E}_i)| \leqslant p$ for all $i \in I$, any $g \in CT(W,\mathcal{E}',p,p)$ has an extension to a $g_1 \in CT(X,\mathcal{E},p,p)$. By using the restriction of ϕ to the minimal ideal $CT(X,\mathcal{E},p,p)$ and noting that $CT(W,\mathcal{E}',p,p)$ is again a factor semigroup of $CT(X,\mathcal{E},p,p)$, it follows that ψ on $CT(W,\mathcal{E}',p,p)$ given by

$$\psi(f|_W) = \phi(f)|_W, \text{ for } f \in CT(X,\mathcal{E},p,p)$$

is a well-defined automorphism, as in Section 3.3.

Let $\{h_\alpha\}_{\alpha\in\Omega}$ be a compatible system associated with ψ, and now consider ψ to be extended (as in Proposition 3.2.40) to Q. We show

$\phi(f)|_W = \psi(f|_W)$ for all $f \in S$, noting that now $f|_W \in Q$. Suppose $R(f) \in M_\alpha$ and $w \in W$ with $w = \phi(\ell)(v)$, for some $\ell \in CT(X,\mathcal{E},p,p)$ with $R(\ell) \in M_\beta$, say, and $v \in W$.

$$\begin{aligned}
\text{Then}\quad \phi(f)(w) &= \phi(f)\phi(\ell)(v) \\
&= \phi(f\ell)(v) \\
&= \psi(f\ell|_W)(v) && \text{as } f\ell \in CT(X,\mathcal{E},p,p), \\
&= h_\alpha f\ell h_\alpha^{-1}(v) && \text{as } R(f\ell) \in M_\alpha, \\
&= (h_\alpha f h_\alpha^{-1})(h_\beta \ell h_\beta^{-1})(v) && \text{(Remark 3.2.26)}, \\
&= (h_\alpha f h_\alpha^{-1})\phi(\ell)(v) && \text{as } \ell \in CT(X,\mathcal{E},p,p), \\
&= \psi(f|_W)(w) && \text{as } R(f|_W) \in M_\alpha .
\end{aligned}$$

Thus $\phi(f)|_W = \psi(f|_W)$ for all f in S.

For the uniqueness of ψ, note that any other automorphism on Q satisfying the conditions of the lemma would have to agree with ψ on $CT(W,\mathcal{E}',p,p)$ and hence on Q, by the extension uniqueness result (Proposition 3.2.40). □

We have already observed that the C_q sets of S are the same as those of Q. Our next lemma is a convenient surprise, easing the path to a description of ϕ.

<u>LEMMA 3.4.6.</u> The order-automorphism H_1 of ϕ is identical to the order-automorphism H of ψ.

<u>Proof.</u> Take $A \in C_q$ and $B \subseteq A$ with $B \in C_p$ and $|A - B| = p$. Choose $g \in Q$ with $R(g) = B$ and extend g to $g_1 \in S$ with $R(g_1) \subseteq A$. Then

$$\begin{aligned}
H(B) &= R(\psi(g)) && \text{from the definition of } H, \\
&= R(\phi(g_1)|_W) && \text{the property of } \psi, \\
&\subseteq R(\phi(g_1)) \\
&\subseteq H_1(A) && \text{from the definition of } H_1 .
\end{aligned}$$

Fixing $A \in C_q$, and taking all unions now over $\{B \in C_p \mid B \subseteq A$ and $|A - B| = p\}$ we have $A = \cup B$, so

$$H(A) = h_A(A) = h_A(\cup B) = \cup h_A(B) = \cup H(B)$$

using Lemmas 3.1.12 and 3.1.13. Thus

$$H(A) = \cup H(B) \subseteq H_1(A).$$

Similarly, ϕ^{-1} on S, associated with H_1^{-1}, yields ψ^{-1} on Q, associated with H^{-1}, so $H^{-1}(A) \subseteq H_1^{-1}(A)$ for all $A \in \mathcal{C}_q$. Thus

$$A = H^{-1}(H(A)) \subseteq H_1^{-1}(H(A)) \subseteq H_1^{-1}(H_1(A)) = A, \quad \text{so} \quad H_1 = H. \qquad \square$$

COROLLARY 3.4.7. For all $A \in \mathcal{M}_\alpha$, $H_1(A) = H(A) = h_\alpha(A)$.

Proof. This follows from Lemma 3.4.6, Definition 3.1.17 and Theorem 3.2.2. □

We conclude our preliminaries with a lemma whose proof is exactly as in the proofs of Proposition 3.3.8 and Theorem 3.3.3.

LEMMA 3.4.8. Given $\phi \in$ Aut S there exists a unique $z \in G_{\&}$ which lifts $\tilde{\psi} \in G_{\&}$, and satisfies

$$\pi(\phi(f)) = z(\pi(f)), \text{ for all } f \text{ in } S. \qquad \square$$

REMARK 3.4.9. Denote by v the permutation of I induced by z, so that $z(\&_i) = \&_{v(i)}$ and $\phi(T_i) = T_{v(i)}$. Just as previously ϕ induced a permutation of the range families $\{\mathcal{M}_\alpha\}_{\alpha\in\Omega}$ and the right ideals $\{I_\alpha\}_{\alpha\in\Omega}$ (Proposition 3.2.21), so now ϕ induces a permutation of equivalences $\{\&_i\}_{i\in I}$ and left ideals $\{T_i\}_{i\in I}$. □

Proof of Theorem 3.4.4. Given $\phi \in$ Aut S we have $\psi \in$ Aut Q and $z \in G_{\&}$ satisfying (i) and (ii) as in Lemmas 3.4.5 and 3.4.8. We proceed to the construction of the y_i bijections. Let $f \in I_\alpha \subseteq S$. Then using Corollary 3.4.7 we have

$$R(\phi(f)) = H_1(R(f)) = h_\alpha(R(f)), \quad \text{and}$$

$$R(\phi(f)|_W) = R(\psi(f|_W)) = H(R(f|_W)) = h_\alpha(R(f|_W)).$$

Thus $$\phi(f)(X) - \phi(f)(W) = h_\alpha(f(X)) - h_\alpha(f(W)) = h_\alpha(f(X) - f(W)).$$

Since $|\mathcal{B}(\pi(f))| = |f(X) - f(W)|$ and h_α is a bijection, we conclude that $\pi(f)$ has as many bad classes as does $\pi(\phi(f))$. Moreover, a correspondence between these classes is suggested: define $y_i : \mathcal{B}(\&_i) \to \mathcal{B}(z(\&_i))$ via

$$y_i(C) = D \text{ where, for } f \in T_i \cap I_\alpha \subseteq S,\ \phi(f)(D) = h_\alpha f(C).$$

Certainly y_i is a bijection, but we must confirm that it does not depend on the choice of $f \in T_i$. Suppose $g \in T_i \cap I_\beta$ and initially that $g = kf$ for some $k \in I_\gamma$, say. Then

$$\begin{aligned}
\phi(g)(D) &= \phi(k)\phi(f)(D) \\
&= \phi(k)h_\alpha f(C) \\
&= (h_\gamma k h_\gamma^{-1})(h_\alpha f(C)) && \text{Lemma 3.4.5 and Theorem 3.2.2,} \\
&= h_\gamma kf(C) && \text{Remark 3.2.26,} \\
&= h_\gamma g(C) \\
&= h_\beta g(C) && \text{since } g \in I_\beta \cap I_\gamma, \text{ and } \{h_\alpha\}_{\alpha\in\Omega} \text{ is a}
\end{aligned}$$

compatible system.

Generally, since $\pi(f) = \pi(g)$, $rf = sg$ for some $r,s \in S$ (Lemma 3.3.7(2)). If $rf \in I_\gamma$, we have from above that $\phi(rf)(D) = h_\gamma rf(C)$, whence $\phi(sg)(D) = h_\gamma sg(C)$. Now $\phi(g)(D') = h_\beta g(C)$, for some $D' \in \mathcal{B}(z(\&_i))$ so $\phi(sg)(D') = h_\gamma sg(C)$, again using the above special case. Thus $\phi(sg)(D) = \phi(sg)(D')$ and hence $D = D'$.

We conclude that y_i is a well-defined bijection from $\mathcal{B}(\&_i)$ to $\mathcal{B}(z(\&_i))$ such that for $f \in T_i \cap I_\alpha$, $\phi(f)(D) = h_\alpha f y_i^{-1}(D)$ for all $D \in \mathcal{B}(z(\&_i))$. Lemma 3.4.5 ensures that ψ is unique, while the uniqueness of z and the y_i maps follows from their definition and the uniqueness of compatible systems mentioned in Remark 3.2.32.

For the converse, given $S = CT(X,\&,p,q)$, and the associated triple of $\psi \in \text{Aut } Q$ (determined by the compatible system $\{h_\alpha\}_{\alpha\in\Omega}$),

$z \in G_{\&}$ lifting $\tilde{\psi}$ and $\{y_i\}_{i \in J}$, with y_i a bijection from $B(\&_i)$ to $B(z(\&_i))$, define ϕ from S to S as follows: for $f \in T_i \cap I_\alpha$ let $\phi(f)$ from X to X be the function uniquely determined by the conditions

(i) $\phi(f)|_W = \psi(f|_W) = h_\alpha f h_\alpha^{-1}$,

(ii) $\pi(\phi(f)) = z(\pi(f))$, and

(iii) $\phi(f)(D) = h_\alpha f y_i^{-1}(D)$ for $D \in B(z(\&_i))$,

as stated in the theorem.

Note that since z lifts $\tilde{\psi}$, (i) and (ii) are consistent. If $f \in I_\beta$ also, $h_\alpha f = h_\beta f$ so $h_\alpha f y_i^{-1}(D) = h_\beta f y_i^{-1}(D)$ for $D \in B(z(\&_i))$, so $\phi(f)$ is well-defined. Furthermore, $\pi(\phi(f)) \in \&$ and $R(\phi(f)) = h_\alpha(R(f)) \in C_q$ so $\phi(f) \in S$.

To show that ϕ is a homomorphism, as in the proof of Theorem 3.3.3 we have, for $f,g \in S$,

$$\pi(\phi(g)\phi(f)) = \pi(\phi(gf)) \quad \text{and} \quad \phi(g)\phi(f)|_W = \phi(gf)|_W$$

so $\phi(g)\phi(f)$ and $\phi(gf)$ agree on good classes. Now let $D \in B(z(\&_i))$ and suppose $f \in I_\alpha$, $g \in I_\beta$. Then

$$\begin{aligned}
\phi(g)\phi(f)(D) &= \phi(g)h_\alpha f y_i^{-1}(D) && \text{using (iii)},\\
&= (h_\beta g h_\beta^{-1})(h_\alpha f y_i^{-1}(D)) && \text{using (i), since } h_\alpha f y_i^{-1}(D) \in W,\\
&= h_\beta g f y_i^{-1}(D) &&\\
&= \phi(gf)(D) && \text{using (iii) since } gf \in I_\beta .
\end{aligned}$$

Thus $\phi(g)\phi(f)$ and $\phi(gf)$ agree on bad classes also.

We show ϕ to be a bijection by exhibiting the inverse. Note that if ψ induces the permutation u of Ω (Proposition 3.2.22) then ψ^{-1} induces u^{-1}, while if z induces v on I (Remark 3.4.9) then z^{-1} induces v^{-1}. Consider the triple of $\psi^{-1} \in \text{Aut } Q$ (with associated compatible system $\{h^{-1}_{u^{-1}(\alpha)}\}_{\alpha \in \Omega}$: see Theorem 3.2.2 proof), the lifting z^{-1} of $\widetilde{\psi^{-1}} = (\tilde{\psi})^{-1}$, and the system of bijections $\{y^{-1}_{v^{-1}(i)}\}$, where $y^{-1}_{v^{-1}(i)}$ maps $B(\&_i)$ onto $B(z^{-1}(\&_i))$. As before, construct a homomorphism χ say, of S, such that for all $f \in T_i \cap I_\alpha$,

(i) $\chi(f)|_W = \psi^{-1}(f|_W)$,

(ii) $\pi(\chi(f)) = z^{-1}(\pi(f))$, and

(iii) $\chi(f)(D) = h^{-1}_{u^{-1}(\alpha)}\ f\ y_{v^{-1}(i)}(D)$, for all $D \in B(z^{-1}(\&_i))$.

Then

$$\pi(\chi\phi(f)) = z^{-1}(\pi(\phi(f))) = z^{-1}z(\pi(f)) = \pi(f),$$

$$\chi\phi(f)|_W = \psi^{-1}(\phi(f)|_W) = \psi^{-1}(\psi(f|_W)) = f|_W,$$

and since $\phi(f) \in T_{v(i)} \cap I_{u(\alpha)}$, for all $C \in B(\&_i)$

$$\begin{aligned}\chi(\phi(f))(C) &= h^{-1}_{u^{-1}(u(\alpha))}\ \phi(f)\ y_{v^{-1}(v(i))}\ (C) = h_\alpha^{-1}\ \phi(f)y_i(C)\\ &= h_\alpha^{-1}h_\alpha\ f\ y_i^{-1}y_i(C)\\ &= f(C),\end{aligned}$$

so $\chi\phi$ is the identity on S. Similarly $\phi\chi$ is the identity, so ϕ is invertible.

Note finally that for $g \in I_\alpha$, $R(\phi(g)) = h_\alpha(R(g))$, so ϕ is range-preserving, because by Proposition 3.2.17(2), $R(f) \subseteq R(g)$ implies $f \in I_\alpha$ also. □

With the aid of Theorem 3.4.4 we extend our earlier characterisation of when Aut S = Inn S (Proposition 3.3.12).

<u>DEFINITION 3.4.10</u>. The semigroup $S = CT(X,\&,p,q)$ has the *modified extension property* (c.f. Definition 3.3.11) if for each p.p. $h \in G_W$, lifting $z \in G_\&$ of $\tilde{h}$ with $|B(z(\&_i))| = |B(\&_i)|$ for all $i \in J$, and bijections $y_i : B(\&_i) \to B(z(\&_i))$ for each $i \in J$, there exists a p.p. $k \in G_X$ such that

(i) k extends h

(ii) $z = \tilde{k}$, and

(iii) $y_i(C) = k(C)$, for all $i \in J$ and $C \in B(\&_i)$. □

<u>PROPOSITION 3.4.11</u>. Let $S = CT(X,\&,p,q)$ be an arbitrary Croisot-Teissier semigroup. Then all range-preserving automorphisms of S are inner if

and only if all C_q sets are λ-related and S has the modified extension property.

Proof. Suppose all range-preserving automorphisms of S are inner. Given h,z and $\{y_i\}_{i\in J}$ as specified in the modified extension property, note that $\psi_h \in$ Aut Q (Proposition 3.1.6) and construct ϕ range-preserving in Aut S, according to Theorem 3.4.4. Let $\phi = \phi_k$. We show that k extends h. Let $f \in CT(W,\mathcal{E}',p,p)$. Since $|B(\mathcal{E}_i)| \leqslant p$ for all $i \in I$, there exists an extension $f_1 \in CT(X,\mathcal{E},p,p)$ of f. Then

$$hfh^{-1} = \psi_h(f) = \psi_h(f_1|_W) = \phi(f_1)|_W = (kf_1k^{-1})|_W = k|_W f(k|_W)^{-1} ,$$

since k and k^{-1} leave W invariant. Applying 3.1.6 to the restriction of ψ_h to the minimal ideal $CT(W,\mathcal{E}',p,p)$ we conclude with the aid of Proposition 3.2.40 that $k|_W = h$, so property (i) holds. Now, for any g in S, by Theorem 3.4.4, $\pi(\phi_k(g)) = z(\pi(g))$. Also $\pi(\phi_k(g)) = \{k(C) \mid C \in \pi(g)\} = \tilde{k}(\pi(g))$. That is $z = \tilde{k}$. This ensures property (ii) holds, while (iii) is clear from Theorem 3.4.4. That all C_q sets are λ-related follows as before, provided we use Theorem 3.2.35 and note that if Aut Q $\neq$ Inn Q there exists $\psi \in$ Aut Q such that $\psi|_{CT(W,\mathcal{E}',p,p)}$ is outer (Proposition 3.2.40).

Conversely, let $\phi \in$ Aut S be range-preserving, with associated ψ,z and $\{y_i\}_{i\in J}$ as in Theorem 3.4.4. Then $\psi = \psi_h$ for some p.p. $h \in G_W$. Now $\tilde{\psi} = \tilde{h}$, so z lifts $\tilde{h}$, whence from the modified extension property there exists a p.p. $k \in G_X$ such that (i)-(iii) in Definition 3.4.10 hold, provided $|B(z(\mathcal{E}_i))| = |B(\mathcal{E}_i)|$ for all $i \in J$. Then

$$\pi(\phi(f)) = z(\pi(f)) = \tilde{k}(\pi(f)) = \pi(kfk^{-1}), \quad \text{and}$$
$$\phi(f)|_W = \psi_h(f|_W) = hfh^{-1} = (kfk^{-1})|_W ,$$

for each $f \in S$. Also, for $D \in B(z(\mathcal{E}_i))$ and $f \in T_i \cap I_\alpha$

$$\begin{aligned}\phi(f)(D) &= h_\alpha f y_i^{-1}(D) && \text{(Theorem 3.4.4)}\\ &= hfk^{-1}(D) && \text{(property (iii))}\\ &= kfk^{-1}(D) && (\text{since } k|_W = h)\ ,\end{aligned}$$

so $\phi = \phi_k$. □

<u>COROLLARY 3.4.12</u>. If $\mathcal{B}(\&_i)$ contains classes of differing cardinality, for some $i \in J$, then S has outer automorphisms.

<u>Proof</u>. With $h = \iota_W$, $z = \iota_\&$ and y_i chosens so that $|y_i(C)| \neq |C|$ for some $C \in \mathcal{B}(\&_i)$, condition (iii) which requires that $|y_i(C)| = |k(C)| = |C| (k \in \mathcal{G}_X)$, cannot hold. □

<u>EXAMPLE 3.4.13</u>. Consider the example in 3.4.2 of $X = C \dot{\cup} D$ where $|D| = 3$ and D is partitioned as

$$D = \{d_1\} \dot{\cup} \{d_2, d_3\}\ .$$

That is, $\& = \{\&_c \mid c \in C\} \cup \{\&_1\}$ where $\&_c = \{\{c, d_1, d_2, d_3\}, \text{singletons}\}$ and $\&_1 = \{\{d_1\}, \{d_2, d_3\}, \text{singletons}\}$. Then $W = C$, $J = \{1\}$ and $\mathcal{B}(\&_1) = \{\{d_1\}, \{d_2, d_3\}\}$. Since $\&_1$ has bad classes of different size, we know from Corollary 3.4.12 that S has an outer automorphism. We construct one as follows. Let ϕ be the range-preserving automorphism of S associated with $\psi = \iota_Q$, $z = \iota_\&$ and $\{y_1\}$ where $y_1 : \mathcal{B}(\&_1) \to \mathcal{B}(\&_1)$ is the transposition of $\{d_1\}$ and $\{d_2, d_3\}$, that is

$$\phi(f)|_W = f|_W \quad , \quad \pi(\phi(f)) = \pi(f)\ ,$$

and if $\pi(f) = \&_1$ then

$$\phi(f)(d_1) = f(d_2) \quad , \quad \phi(f)(d_2) = f(d_1)\ .$$

As promised, ϕ is outer because y_1 clearly cannot be induced by a set map k of X. □

3.5. AUTOMORPHISMS IN GENERAL (VIA S^2)

In Section 3.4 we gave a complete description of range-preserving automorphisms of a general Croisot-Teissier semigroup $S = CT(X,\mathcal{E},p,q)$. If $p > q$ then in general there exist automorphisms ϕ which are not range-preserving. At first glance it would appear that we cannot produce an easy description of such ϕ, because our earlier basic technique, of looking at the order-automorphism $H : C_q \to C_q$ of ϕ and the associated h_A maps, no longer works. Fortunately it turns out that these ϕ are never far from being range-preserving, and a complete description is available once we handle a very special case, namely when ϕ is an "η-stabilizing automorphism" (for a certain natural congruence η on S). Here this means $\phi(f)$ and f have the same partition, differ only on "bad" classes, and even then their values are still ρ-equivalent. These η-stabilizing automorphisms are easily characterized (Corollary 3.5.12). The main theorem of this section is then:

<u>THEOREM 3.5.1</u>. Let ϕ be an automorphism of an arbitrary Croisot-Teissier semigroup $CT(X,\mathcal{E},p,q)$. Then ϕ can be expressed uniquely as a product

$$\phi = \phi_1 \circ \phi_2$$

of a range-preserving automorphism ϕ_1 and an η-stabilizing automorphism ϕ_2. Moreover, ϕ is range-preserving if and only if $\phi_2 = \iota$. □

The key ideas in obtaining this factorization are the use of the ideal S^2 as an arbiter of η-stabilizing automorphisms, and the observation that automorphisms of S^2 are always range-preserving. As a corollary to our main theorem, we are able to tell exactly when all automorphisms of S are range-preserving (Theorem 3.5.14). This is so if and only if either $S = S^2$, or W is reduced, or W provides $\mathcal{E}$ SEE ERRATA cross-sections. We find these three (independent) conditions surprising.

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We begin by examing the ideal $S^2 = \{fg \mid f,g \in S\}$. Given a C_q set A and a partition $B \in \&$, let us define the *defect of* A *in* B *by*

(3.5.1) defect(A,B) = number of classes of B which do not meet A.

Note that defect (A,B) $\geqslant$ q because C_q sets are already q-short of some containing w.s. set. As a semigroup, S^2 is itself Croisot-Teissier:

PROPOSITION 3.5.2. $S^2 = CT(p,r)$ where

$$r = \min\{\text{defect}(A,B) \mid A \in C_q,\ B \in \&\}\ .$$

Proof. If $f \in S^2$, say $f = t\ell$ with $t,\ell \in S$, then

$$|R(t) - R(f)| = \text{defect}(R(\ell),\ \pi(t)) \geqslant r\ .$$

Hence $f \in CT(p,r)$ and so $S^2 \subseteq CT(p,r)$.

For the reverse containment, assume $f \in CT(p,r)$. Choose $C \in C_q$ with $R(f) \subseteq C$ and $|C - R(f)| = r$. Choose $A \in C_q$ and $B \in \&$ such that

$$\text{defect}(A,B) = r\ .$$

Since defect(A,B) = $|C - R(f)|$, we can choose $t \in S$ with $\pi(t) = B$, $t(A) = R(f)$, and $R(t) = C$. Note t is 1-1 on A. Now let $\ell : X \to X$ be the function uniquely determined by

$$\pi(\ell) = \pi(f),\ R(\ell) = A,\ t(\ell(x)) = f(x) \quad \forall x \in X.$$

Clearly $\ell \in S$ and $f = t\ell \in S^2$. Hence $CT(p,r) \subseteq S^2$. □

REMARKS 3.5.3.

(1) When $p = q$, $S = S^2$ always holds.

(2) When $p > q$ Mielke ([15] and Remark 3.0.1) has modified the Clifford and Preston definition of Croisot-Teissier semigroups to ensure that they remain simple with a minimal left ideal. The Baer-Levi semigroup BL(p,q) now becomes a special case of these modified Croisot-Teissier semigroups (c.f. Example 3.1.3). For such modified Croisot-Teissier semigroups on an X,& pair to be non-empty, there must exist a well-separated set with defect at most q in *all* partitions

$B \in \mathcal{E}$ (termed q-well-separated in [15]). We note that the original (richer) Clifford and Preston Croisot-Teissier semigroup S on such an $X, \mathcal{E}$ pair (which we have used throughout) would satisfy $S = S^2$ by 3.5.2 $(r = q)$, though it may not be simple. In general, however, $S \neq S^2$ when no q-well-separated sets are present. The next example shows that S^2 can be arbitrary $CT(p,r)$. □

EXAMPLE 3.5.4. Given any infinite cardinals $p \geqslant r \geqslant q$, there is a Croisot-Teissier semigroup $S = CT(X,\mathcal{E},p,q)$ for which $S^2 = CT(p,r)$: let $X = A \dot{\cup} B \dot{\cup} C$ where $|A| = |B| = p$, $|C| = r$, and consider $\mathcal{E} = \{\mathcal{E}_{b,c} \mid b \in B,\ c \in C\}$ where $\mathcal{E}_{b,c}$ is the partition of X

$$\{\{b,c\},\ \text{singletons}\}\ .$$

Then there are two maximal w.s. sets, namely $A \cup B$ and $A \cup C$. The first has defect r in any partition, while the second has defect p. Hence

$$r = \min\{\text{defect}(D,\mathcal{B}) \mid D \in \mathcal{C}_q,\ \mathcal{B} \in \mathcal{E}\}$$

and so $S^2 = CT(p,r)$ by Proposition 3.5.2. Notice $X = W$ here. □

We now establish that automorphisms in the case $S = S^2$ are range-preserving, by algebraically characterizing range inclusion in this case.

PROPOSITION 3.5.5. Assume $S = S^2$.

(1) For $f,g \in S$, $R(f) \subseteq R(g)$ if and only if there exist $t,f_1,g_1,m \in S$ such that

$$f = tf_1$$
$$g = tg_1$$
$$mg_1 = g_1$$

and for all $k \in S$,

$$kg_1 = g_1 \quad \text{implies} \quad kf_1 = f_1\ .$$

(2) Automorphisms of S are range-preserving.

Proof.

(1) Since $S = S^2$, by Proposition 3.5.2 there exists r with $p \geqslant r \geqslant q$ such that $S = CT(p,r)$ and

$$r = \text{defect}(A,\mathcal{B})$$

for some $A \in \mathcal{C}_q$ and $\mathcal{B} \in \mathcal{E}$.

Assume $R(f) \subseteq R(g)$. Following the same lines as the proof of 3.5.2, we can choose $t \in S$ with $\pi(t) = \mathcal{B}$ and $t(A) = R(g)$. Let $B \subseteq A$ be such that $t(B) = R(f)$. Since t maps B bijectively onto $R(f)$, there is a function $f_1 : X \to X$ with $\pi(f_1) = \pi(f)$, $R(f_1) = B$ and $t(f_1(x)) = f(x)$ for all $x \in X$. Clearly $f_1 \in S$ and $f = tf_1$. Similarly there exists $g_1 \in S$ with $R(g_1) = A$ and $g = tg_1$.

Observe that $\mathcal{C}_q = \mathcal{C}_r$ because $CT(p,q) = S = CT(p,r)$. Hence as $A \in \mathcal{C}_q$, there exists $C \in \mathcal{C}_q$ with $A \subseteq C$ and $|C - A| = r$. Further, as $\text{defect}(A,\mathcal{B}) = r = |C - A|$, we can choose $m \in S$ with $\pi(m) = \mathcal{B}$, $R(m) = C$, and $m|_A = \iota_A$. Clearly $mg_1 = g_1$ because $R(g_1) = A$. Also $R(f_1) \subseteq R(g_1)$, whence for any $k \in S$, $kg_1 = g_1$ implies $kf_1 = f_1$.

Conversely assume the stated conditions on t, f_1, g_1, m. Our proof here proceeds independently of the assumption $S = S^2$, and is in essence the proof in Proposition 3.1.8. Suppose by way of a contradiction that $R(f_1) \not\subseteq R(g_1)$. Choose $x \in R(f_1) - R(g_1)$. Let $B = R(g_1)$, $C = R(m)$, $\mathcal{B} = \pi(m)$ and $r = \text{defect}(B,\mathcal{B})$. Choose a cross-section $T = B \dot\cup Y$ for $\mathcal{B}$ (so $|Y| = r$). Now

$$\begin{aligned} |C - B| &= |R(m) - R(g_1)| \\ &= |R(m) - R(mg_1)| \\ &= \text{defect}(R(g_1),\pi(m)) \\ &= r \ (\geqslant q). \end{aligned}$$

Write $C - B = D_1 \dot\cup D_2$ where $|D_1| = |D_2| = r$ and $x \notin D_1$. Choose a bijection $k_1 : Y \to D_1$ and define $k : X \to X$ by declaring $\pi(k) = \mathcal{B}$ and letting k act on the cross-section T as k_1 on Y and the identity on B. Then $k \in S$ because $R(k) = B \cup D_1 \subseteq C$. Certainly $kg_1 = g_1$ because $R(g_1) = B$, but $kf_1 \neq f_1$ because $x \in R(f_1) - R(k)$. This contradiction proves $R(f_1) \subseteq R(g_1)$, and so $R(f) = R(tf_1) \subseteq R(tg_1) = R(g)$, giving $R(f) \subseteq R(g)$.

(2) This is immediate from (1). □

REMARK. Proposition 3.5.5(2) generalizes Corollary 3.1.9 because, by Proposition 3.5.2, $S = S^2$ when $p = q$. However the condition used to characterize range inclusion when $p = q$ (see 3.1.8) does not work in the general case $S = S^2$. □

An important link between general automorphisms and the range-preserving ones is provided in:

COROLLARY 3.5.6. Let $S = CT(X,\mathcal{E},p,q)$ and $T = S^2$. Then $T = CT(X,\mathcal{E},p,r)$ for some $p \geqslant r \geqslant q$, and $T = T^2$. Furthermore each automorphism ϕ of S induces (by its restriction) a range-preserving automorphism of T.

Proof. We know from Proposition 3.5.2 that $T = CT(p,r)$ for some $p \geqslant r \geqslant q$. By the proof of Proposition 3.5.5, for $f \in S^2$ there exist $t,f_1,m \in S$ such that $f = tf_1$ and $mf_1 = f_1$. Then $f = tm^nf_1$ for all positive n, showing $S^n = S^2$ for all $n \geqslant 2$. In particular $T^2 = T$. (This also follows directly from 3.5.2). Clearly $\phi(S^2) = S^2$, whence $\phi|_T \in \text{Aut } T$ and it is range-preserving by Proposition 3.5.5(2). □

We begin the second part of our attack on automorphisms which are not range-preserving by considering a natural congruence η on S.

DEFINITION 3.5.7. Let η be the relation on S given by:

$$f \;\eta\; g \quad \text{if} \quad f|_W = g|_W \quad \text{and} \quad f(x) \;\rho\; g(x), \text{ for all } x \in X. \qquad \square$$

Thus f is η-related to g if and only if f and g have the same partition, agree on (good) classes meeting W, and have ρ-equivalent values on the other (bad) classes. Using arguments in Lemmas 3.2.7(2) and 3.3.4(2), we quickly obtain the following algebraic formulation of η.

LEMMA 3.5.8. For $f,g \in S$, $f \,\eta\, g$ if and only if $f\ell = g\ell$ and $\ell f = \ell g$ for all $\ell \in S$. □

An easy consequence of 3.5.8 is:

LEMMA 3.5.9.

(1) η is a congruence on S;

(2) η is preserved by automorphisms (that is, $f \,\eta\, g$ implies $\phi(f) \,\eta\, \phi(g)$). □

DEFINITION 3.5.10. A general function $\phi : S \to S$ is called an *η-stabilizing function* if $\phi(f) \,\eta\, f$ for all $f \in S$. □

The ideal S^2 now enters in its second role, as an arbiter of η-stabilizing automorphisms:

PROPOSITION 3.5.11. Let $\phi : S \to S$ be a general function. Then ϕ is an η-stabilizing automorphism if and only if ϕ is an η-stabilizing bijection which fixes S^2 element-wise.

Proof. Suppose ϕ is an η-stabilizing automorphism and let $f \in S^2$, say $f = kg$. By Lemma 3.5.8

$$\begin{aligned} \phi(f) = \phi(k)\phi(g) &= k\phi(g) && \text{since } \phi(k) \,\eta\, k \\ &= kg && \text{since } \phi(g) \,\eta\, g \\ &= f\,. \end{aligned}$$

Thus ϕ fixes S^2.

Conversely assume ϕ is an η-stabilizing bijection which fixes S^2.

For $f,g \in S$, $\phi(fg) = fg$ because ϕ fixes S^2, while $\phi(f)\phi(g) = f\phi(g) = fg$ by Lemma 3.5.8. Thus $\phi(fg) = \phi(f)\phi(g)$ and so ϕ is an η-stabilizing automorphism. □

We now have the following simple description of η-stabilizing automorphisms:

COROLLARY 3.5.12. An η-stabilizing automorphism ϕ of S determines, and is uniquely determined by, an (independent) family $\{m_C\}$ of bijections

$$m_C : C \to C,$$

one for each η-class C in $S - S^2$ (note S^2 is η-closed). For a given class C, say the class of f whose partition has $\{B_j\}$ as its bad classes,

$$G_C \cong G_{\prod_j U_j}$$

SEE ERRATA

where U_j is the ρ-class of $f(B_j)$.

Proof. Observe that S^2 is η-closed, for if $f \in S^2$ and $g \in S$ such that $f \eta g$, then $f\ell = g\ell$ for all $\ell \in S$ (Lemma 3.5.8) implies that $g \in CT(p,r) = S^2$ (Proposition 3.5.2). By Proposition 3.5.11 ϕ is an η-stabilizing automorphism if and only if ϕ fixes each η-class and $\phi|_{S^2} = \iota_{S^2}$. Since ϕ is a bijection, $\phi|_C \in G_C$, for each η-class C in $S - S^2$. The first statement follows.

For the second statement observe that any $g \in C$ is completely determined by f and the values $g(B_j) \in U_j$, while each element of $\prod_j U_j$, together with f, determines some $g \in C$. Hence there is a bijection between C and $\prod_j U_j$, thus $|C| = |\prod_j U_j|$, and so

$$G_C \cong G_{\prod_j U_j}.$$ □

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PROPOSITION 3.5.13. An η-stabilizing automorphism ϕ of S is range-preserving if and only if $\phi = \iota$.

Proof. Suppose $\phi \neq \iota$ is an η-stabilizing automorphism. We show ϕ

is not range-preserving.

Choose $f \in S$ with $\phi(f) \neq f$. Inasmuch as $\phi(f)\ \eta\ f$ but $\phi(f) \neq f$, there must exist $C \in \mathcal{B}(\pi(f))$ with $\phi(f)(C) \neq f(C)$. Fix a (good) class D of $\pi(f)$ which meets W. Define $g \in S$ by declaring $\pi(g) = \pi(f)$,

$$g(D) = f(C)$$

$$g(C) = f(D)$$

$$g(E) = f(E) \quad \text{for all other classes } E \neq C,D\,.$$

Clearly $R(g) = R(f)$. On the other hand, since D is a good class of $\pi(g)$ and $\phi(g)\ \eta\ g$,

$$\phi(g)(D) = g(D) = f(C).$$

Thus $f(C) \in R(\phi(g))$, whereas $f(C) \notin R(\phi(f))$ because $\phi(f)(C)\ \rho\ f(C)$ but $\phi(f)(C) \neq f(C)$ (and $R(\phi(f))$ is well-separated). Hence $R(\phi(g)) \neq R(\phi(f))$, showing ϕ is not range-preserving. □

<u>Proof of Theorem 3.5.1</u>. Let $\phi \in \operatorname{Aut} S$. By Corollary 3.5.6 we know $S^2 = CT(X,\mathcal{E},p,r)$ for some $p \geqslant r \geqslant q$, and $\phi|_{S^2}$ is a range-preserving automorphism of $CT(X,\mathcal{E},p,r)$. Hence by Theorem 3.4.4, there exist an automorphism ψ of $CT(W,\mathcal{E}',p,r)$, $z \in G_{\mathcal{E}}$ and bijections $y_i : \mathcal{B}(\mathcal{E}_i) \to \mathcal{B}(z(\mathcal{E}_i))$ such that, for any compatible system $\{h_\alpha\}_{\alpha\in\Omega}$ associated with ψ and for all $f \in S^2$,

(i) $\phi(f)|_W = \psi(f|_W)$

(ii) z is a lifting of $\widetilde{\psi}$ and $\pi(\phi(f)) = z(\pi(f))$

(iii) $\phi(f)(D) = h_\alpha f y_i^{-1}(D) \quad \forall f \in I_\alpha \cap T_i, \quad \forall D \in \mathcal{B}(z(\mathcal{E}_i))$.

By Proposition 3.2.40, ψ has a unique extension to an automorphism ψ_1 of $CT(W,\mathcal{E}',p,q)$ and $\{h_\alpha\}_{\alpha\in\Omega}$ remains a compatible system for ψ_1. Notice that $\widetilde{\psi}_1 = \widetilde{\psi}$ by Proposition 3.3.8. By the converse of Theorem 3.4.4, there is a unique range-preserving automorphism ϕ_1 of S associated with the system $\psi_1, z, \{y_i\}$. Since ψ_1 extends ψ, we note that ϕ_1 and ϕ agree on S^2.

<u>Claim</u>: $\phi(f) \ \eta \ \phi_1(f)$, for all $f \in S$.

We establish this using a variation on Lemma 3.5.8, namely:

$$f \ \eta \ g \text{ if and only if } f\ell = g\ell \text{ and } \ell f = \ell g, \text{ for all } \ell \in S^2.$$

This is proved in a similar fashion to 3.5.8 by observing that $S^2 \supseteq CT(p,p)$, and that appropriate ℓ needed for the "if" part of 3.5.8 can be chosen in $CT(p,p)$. Now using the fact that ϕ and ϕ_1 agree on S^2, we have for all $s \in S^2$

$$\phi(s)\phi(f) = \phi(sf) = \phi_1(sf) = \phi_1(s)\phi_1(f) = \phi(s)\phi_1(f)$$

whence $\quad \ell\phi(f) = \ell\phi_1(f)$, for all $\ell \in S^2$.

Similarly $\quad \phi(f)\ell = \phi_1(f)\ell$, for all $\ell \in S^2$. Thus $\phi(f) \ \eta \ \phi_1(f)$ as claimed.

Now set $\phi_2 = \phi_1^{-1}\phi$. By Lemma 3.5.9(2), $\phi(f) \ \eta \ \phi_1(f)$ implies $\phi_1^{-1}(\phi(f)) \ \eta \ \phi_1^{-1}(\phi_1(f))$, that is $\phi_2(f) \ \eta \ f$, for all $f \in S$. Hence ϕ_2 is an η-stabilizing automorphism, and we have the desired factorization $\phi = \phi_1 \circ \phi_2$.

To show the uniqueness of ϕ_1 and ϕ_2 assume there is also a range-preserving automorphism ϕ_1' and a η-stabilizing automorphism ϕ_2' such that

$$\phi_1'\phi_2' = \phi = \phi_1\phi_2, \quad \text{or}$$
$$\phi_1^{-1}\phi_1' = \phi_2(\phi_2')^{-1}.$$

The range-preserving automorphisms form a subgroup of Aut S which, by Proposition 3.5.13, intersects trivially with the subgroup of η-stabilizing automorphisms. Hence

$$\phi_1^{-1}\phi_1' = \phi_2(\phi_2')^{-1} = \iota_S,$$

and the uniqueness of ϕ_1 and ϕ_2 follows. To show the final statement of the Theorem, let $\phi = \phi_1\phi_2$ be range-preserving. Then

$$\phi_1^{-1}\phi = \phi_1^{-1}\phi_1\phi_2 = \phi_2$$

is range-preserving, so $\phi_2 = \iota_S$ by Proposition 3.5.13. □

REMARK. Aut S is actually a split extension of the group G_2 of η-stabilizing automorphisms by the group G_1 of range-preserving automorphisms. Indeed, as we showed in Theorem 3.5.1

$$\text{Aut } S = G_1 G_2 = \{\phi_1\phi_2 \mid \phi_1 \in G_1, \phi_2 \in G_2\} \quad \text{and}$$
$$G_1 \cap G_2 = \{\iota_S\} \quad \text{(Proposition 3.5.13)}.$$

Thus we only have to show that G_2 is a normal subgroup of Aut S. Let $\psi \in G_2$ and $\phi \in$ Aut S. We show $\phi\psi\phi^{-1} \in G_2$. For each $f \in S$, $\psi\phi^{-1}(f)\ \eta\ \phi^{-1}(f)$, and so by Lemma 3.5.9(2)

$$\phi\psi\phi^{-1}(f)\ \eta\ f\ ,$$

that is $\phi\psi\phi^{-1} \in G_2$, as required. □

An unexpected consequence of Theorem 3.5.1 is the following:

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THEOREM 3.5.14. All automorphisms of $S = CT(X,\&,p,q)$ are range-preserving if and only if

either $S = S^2$

or W is reduced SEE ERRATA

or W provides & cross-sections.

A REMARK 3.5.15. The second and third conditions can also be characterized algebraically in terms of S. They are respectively equivalent to the following congruences μ and σ on S being trivial:

$$f\ \mu\ g \quad \text{if} \quad \ell f = \ell g \quad \forall \ell \in S \quad \text{(see 3.2.8)}$$
$$f\ \sigma\ g \quad \text{if} \quad f\ell = g\ell \quad \forall \ell \in S \quad \text{and} \quad rf = sg \text{ for some } r,s \in S.$$

Notice that $\sigma = \nu \cap \pi$ where ν is the congruence mentioned in 3.3.5, and π is the congruence "has the same partition as" (see 3.3.7). Note also $\eta = \mu \cap \nu$. The three conditions in Theorem 3.5.14 are quite independent! □

Proof. By 3.5.1, 3.5.2 and 3.5.12, there exists an automorphism which is not range-preserving

$\Leftrightarrow$ there exists a non-identity η-stabilizing automorphism

$\Leftrightarrow$ there exists a non-trivial η-class C in $S - S^2$

$\Leftrightarrow$ there exists $f \in S - S^2$ and $B \in \mathcal{B}(\pi(f))$ such that the ρ-class of $f(B)$ is non-trivial

$\Leftrightarrow$ $S \neq S^2$, W fails to provide $\mathcal{E}$ cross-sections, and W is not reduced. SEE ERRATA $\square$

REMARKS 3.5.16.

(1) It follows from Theorem 3.5.1, Proposition 3.2.17(1) and Theorem 3.4.4, that for $\phi \in \text{Aut } S$ and $f,g \in S$

$$R(f)^c \subseteq R(g)^c \quad \text{implies} \quad R(\phi(f))^c \subseteq R(\phi(g))^c,$$

where closure is with respect to ρ. This can also be proved directly by showing that the first inclusion is characterized algebraically by:

$$(3.5.2) \qquad rg = sg \quad \text{implies} \quad rf = sf, \quad \forall r,s \in S.$$

Thus an automorphism is never very far from being range-preserving.

We show now that $R(f)^c \subseteq R(g)^c$ iff (3.5.2) holds.

If $R(f)^c \subseteq R(g)^c$, then $rg = sg$ means that r and s are identical on $R(g)$, so also on $R(g)^c$ and hence on $R(f)^c$, thus $rf = sf$.

For the converse assume that $v \in R(f)^c - R(g)^c$. Fix an $A \in \mathcal{E}$. If there exists $w \in R(g)$ with $v\!\!/Aw$, then this w is unique, because $R(g)$ is w.s., and we can choose $B \in \mathcal{E}$ such that v and w are not B-equivalent. Otherwise let $B = A$. Choose a w.s. A and let $A = B \dot{\cup} C \dot{\cup} D$ with $|B| = |C| = |D| = p$. Let $U,V \subseteq X$ be such that $R(g) \dot{\cup} U$ and $R(g) \dot{\cup} V$ are cross-sections of A and B respectively. Choose a bijection

$$k : R(g) \to B,$$

and partial 1-1 transformations

$$\ell : U \to C \quad \text{and} \quad m : V \to D.$$

Define maps $r : X \to X$, $s : X \to X$ by letting $\pi(r) = A$, $\pi(s) = B$ and for any $x \in X$

$$r(x) = \begin{cases} k(y), & y \in R(g), \quad x\,A\,y, \\ \ell(z), & z \in U \quad , \quad x\,A\,z, \end{cases}$$

$$s(x) = \begin{cases} k(y), & y \in R(g), \quad x\,B\,y, \\ m(z), & z \in V \quad , \quad x\,B\,z. \end{cases}$$

Certainly, $r, s \in S$ and $rg = sg$. We show $r(v) \neq s(v)$. If $B = A$, then there are no elements A-equivalent to v in $R(g)$, so $r(v) \in C$, $s(v) \in D$. If $B \neq A$, then vAw, $w \in R(g)$, so

$$r(v) = r(w) = s(w) \neq s(v) \;,$$

since v and w are not B-equivalent. Thus $rf \neq sf$.

(2) We have algebraic characterizations of range inclusion when $S = S^2$ (Proposition 3.5.5(1)) or W is reduced (see (1)). Such a SEE ERRATA characterization has eluded us in the remaining case when W provides & cross-sections (or even when $W = X$). □

We began in Section 3.1 with conditions which yielded the property that all automorphisms were inner, and have traced the recession of this property through 3.2.35, 3.3.12 and 3.4.11. A feature of earlier results ([19], [5], [8], [11]) on automorphisms of transformation semigroups was that all automorphisms were inner. Corollary 3.5.18 (below) reveals that amongst Croisot-Teissier semigroups those with Aut S = Inn S are substantially restricted. While the property of all automorphisms being range-preserving is algebraically characterized in terms of S (Remark 3.5.15), this is not the case for the property of SEE ERRATA all automorphisms being inner, because, as Example 3.5.17 reveals, the latter property is not preserved under isomorphism. For this reason we now consider the range-preserving property of equal interest.

<u>EXAMPLE 3.5.17</u>. Let $\{Y_j\}_{j\in\Delta}$ be a countable collection of disjoint countable similarly well-ordered sets. Let $X = \dot{\bigcup\limits_{j\in\Delta}} Y_j \,\dot{\cup}\, \{x\}$, y_{ij} be the i-th element of Y_j and for each i, $D_i = \{y_{ij} \mid j \in \Delta\}$. Choose a $y \in X - (D_1 \,\dot{\cup}\, \{x\})$ and let

$$\mathcal{E}_1 = \{D_1, \{x,y\}, \text{singletons}\},$$

$$\mathcal{E}_i = \{D_i \,\dot{\cup}\, \{x\}, \text{singletons}\},\ i = 2,3,\ldots\ ,$$

and $\mathcal{E} = \{\mathcal{E}_i \mid i \in I\}$, where $I = \mathbb{N}$. Then $A \subseteq X$ is a w.s. set iff $|A| = \aleph_0$, $x \notin A$ and $|A \cap D_i| \leqslant 1$, each $i \in I$. Clearly $W = X - \{x\}$ and $\rho = \iota_X$. Thus by Theorem 3.1.2 all automorphisms of $Q = CT(W,\mathcal{E}',\aleph_0,\aleph_0)$ are inner. However, $S = CT(X,\mathcal{E},\aleph_0,\aleph_0)$ has outer automorphisms. Indeed, in view of Proposition 3.3.12 it is sufficient to find a p.p. $h \in G_W$ and lifting $z \in G_{\mathcal{E}}$ of $\tilde{h}$ for which there is no p.p. $k \in G_X$ extending h such that $z = \tilde{k}$. Now, let

$$h \in G_W,\ h(D_1) = D_2,\ h(D_2) = D_1,\ h|_{\bigcup\limits_{i>2} D_i} = \iota\ ,$$

then

$$\tilde{h} \in G_{\mathcal{E}'}\ ,\ \tilde{h}(\mathcal{E}_1') = \mathcal{E}_2'\ ,\ \tilde{h}(\mathcal{E}_2') = \mathcal{E}_1'\ ,\ \tilde{h}|_{\{\mathcal{E}_i'\}_{i>2}} = \iota$$

and

$$z \in G_{\mathcal{E}}\ ,\ z(\mathcal{E}_1) = \mathcal{E}_2\ ,\ z(\mathcal{E}_2) = \mathcal{E}_1\ ,\ z|_{\{\mathcal{E}_i\}_{i>2}} = \iota\ .$$

So if h has a p.p. extension $k \in G_X$, then $z = \tilde{k}$ and hence

$$\tilde{k}(\mathcal{E}_1) = \{k(D_1), k(\{x,y\}), \text{singletons}\} = \mathcal{E}_2\ ,$$

a contradiction, because $\mathcal{E}_2$ does not contain a doubleton. We conclude that such a k does not exist and so S has outer automorphisms.

Finally we show that $S \cong Q$. Define $\theta : S \to Q$ via $\theta(f) = f|_W$. Lemma 3.3.4 ensures that θ is a homomorphism from S onto Q. Hence we only have to show that θ is 1-1. Indeed, let $f_1, f_2 \in S$ and $\theta(f_1) = \theta(f_2)$. Then

$$f_1|_W = f_2|_W\ ,$$

thus

$$\gamma\pi(f_1) = \pi(f_1|_W) = \pi(f_2|_W) = \gamma\pi(f_2).$$

Since in this example γ is 1-1, we conclude, that $\pi(f_1) = \pi(f_2)$, so $f_1(x) = f_2(x)$ and thus $f_1 = f_2$. □

COROLLARY 3.5.18. Theorem 3.5.14 combined with Proposition 3.4.11 completes the characterization of when Aut S = Inn S in general.

Proof. Simply observe that inner automorphisms are surely range-preserving. □

We present a simple example of an automorphism which is not range-preserving. In view of Theorem 3.5.14, we must ensure that $S \neq S^2$, W is not reduced, and W fails to provide $\mathcal{E}$ cross-sections. SEE ERRATA

EXAMPLE 3.5.19. Fix $p > q$. Let $X = C \dot{\cup} D \dot{\cup} \{b\}$ with $|C| = |D| = p$. Fix $x,y \in C$. Now consider the family of partitions

$$\mathcal{E} = \{\mathcal{E}_{c,d}, \mathcal{E}_{b,c}, \mathcal{E}_{b,d} \mid c \in C, d \in D\}$$

where

$$\mathcal{E}_{c,d} = \{\{c,d\}, \{x,y\}, \text{singletons}\}$$
$$\mathcal{E}_{b,c} = \{\{b,c\}, \{x,y\}, \text{singletons}\}$$
$$\mathcal{E}_{b,d} = \{\{b,d\}, \{x,y\}, \text{singletons}\},$$

and let $S = CT(X,\mathcal{E},p,q)$ be the associated Croisot-Teissier semigroup. Then Y is w.s. iff $Y \subseteq D$ or $Y \subseteq C$ and has no more than one point in common with $\{x,y\}$. Observe that

$S^2 = CT(p,p) \neq S$ (by 3.5.2)

$W = C \cup D \neq X$ SEE ERRATA

$\{x,y\}$ is a non-trivial ρ-class in W SEE ERRATA

$\{b\}$ is a bad class of $\mathcal{E}_{c,d}$.

Let $e : X \to X$ be the transposition (x,y). Now define $\phi : S \to S$ by letting ϕ fix S^2 element-wise, and for $f \notin S^2$ letting

$$\phi(f)(a) = \begin{cases} f(a) \text{ if } a \text{ lies in a good class of } \pi(f) \\ ef(a) \text{ if } a \text{ lies in a bad class of } \pi(f) . \end{cases}$$

Then $\phi \neq \iota$ is an η-stabilizing automorphism (see 3.5.11), and hence is not range-preserving (see 3.5.13). To see this directly, choose $E \subseteq C$ with $|C - E| = q$ and $x \in E$, $y \notin E$. Choose $f,g \in S$ with $R(f) = R(g) = E$, $\pi(f) = \pi(g) = \mathcal{E}_{c,d}$, and

$$f(b) = x$$

$$g(c) = x .$$

Then $R(\phi(f)) = (E - \{x\}) \cup \{y\}$ whereas $R(\phi(g)) = E$. $\square$

3.6. GREEN'S RELATIONS ON $CT(X,\mathcal{E},p,q)$

In this section we establish Green's relations on an arbitrary Croisot-Teissier semigroup. We start by presenting some definitions and preliminary results.

Let S be an arbitrary transformation semigroup without the identity ι_X. Denote by S^1 the semigroup $S \cup \iota_X$. We say [3,p.47] that f and g in S are *$\mathcal{R}$-equivalent*, and write $f\mathcal{R}g$, if they generate the same principal right ideal. That is, if

(3.6.1) $$fS^1 = gS^1, \quad \text{or} \quad f \cup fS = g \cup gS\ .$$

Clearly, $\mathcal{R}$ is an equivalence relation such that

$$f\mathcal{R}g \ \text{ implies } \ sf\mathcal{R}sg\ , \ \text{for all } s \text{ in } S\ .$$

So, $\mathcal{R}$ is a *left congruence*.

Dually we define $f\mathcal{L}g$ to mean

(3.6.2) $$S^1f = S^1g, \quad \text{or} \quad f \cup Sf = g \cup Sg\ .$$

If $f\mathcal{L}g$ we say that f and g are *$\mathcal{L}$-equivalent*. The equivalence $\mathcal{L}$ is a *right congruence* on S, that is

$$f\mathcal{L}g \ \text{ implies } \ fs\mathcal{L}gs\ , \ \text{for all } s \text{ in } S\ .$$

Note that a semigroup S is left [right] simple (that is, contains no proper left [right] ideals) if and only if it consists of a single $\mathcal{L}[\mathcal{R}]$ class [3,p.48].

The intersection of the equivalences $\mathcal{L}$ and $\mathcal{R}$ is an equivalence relation which is denoted by $\mathcal{H}$. It is clear that $\mathcal{H}$ is a left and right congruence, that is, a congruence on S. SEE ERRATA

The equivalences $\mathcal{L}$ and $\mathcal{R}$ commute [3,p.47]. Hence the relation

$$\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$$

is an equivalence relation on X, and is the *join* $\mathcal{L} \vee \mathcal{R}$ of $\mathcal{L}$ and $\mathcal{R}$ [3, Lemma 1.4], that is the equivalence relation generated by $\mathcal{L} \cup \mathcal{R}$.

We say that f and g in S are *$\mathcal{J}$-equivalent* if they generate the same two-sided principal ideal. That is, if

(3.6.3) $S^1fS^1 = S^1gS^1$; or $f \cup fS \cup Sf \cup SfS = g \cup gS \cup Sg \cup SgS$.

A semigroup S is simple (that is contains no proper ideals) if and only if it consists of a single $\mathcal{J}$ class.

Now let S be an arbitrary Croisot-Teissier semigroup, $S = CT(X,\mathcal{E},p,q)$.

<u>PROPOSITION 3.6.1</u>. Given f,g in S,

$$f\mathcal{R}g \text{ iff } f = g .$$

<u>Proof</u>. If f = g then certainly $f\mathcal{R}g$. For the converse assume $f\mathcal{R}g$ and $f \neq g$. It follows from (3.6.1) that there exist s,t in S such that

$$f = gs \text{ and } g = ft .$$

Hence $R(f) \supseteq R(g)$ and $R(g) \supseteq R(f)$, or $R(f) = R(g)$. Let a w.s. A contains R(t) with $A - R(t) \neq \Phi$. Then

$$\begin{aligned} R(f) = f(X) \supseteq f(A) &= f(R(t)) \dot{\cup} f(A-R(t)), \text{ because f is 1-1 on w.s. sets} \\ &= R(g) \dot{\cup} f(A-R(t)) \\ &\supsetneq R(g) . \end{aligned}$$

This contradicts the fact that R(f) = R(g) and allows us to conclude that f = g. □

Recall (see (3.5.1)) that given a C_q set A and a partition $B \in \mathcal{E}$, the defect of A in B is defined by

defect(A,B) = number of classes of B which do not meet A.

For a cardinal r, with $p \geqslant r \geqslant q$, we let

$$R(r) = \{f \in S \mid \text{for some } A \in \mathcal{E}, \text{ defect}(R(f),A)) < r\} .$$

Note that $R(r) \subseteq S - CT(p,r)$, $R(r)$ is not necessarily non-empty for every r and $R(q) = \Phi$. For a function f in S, whether f is in $R(r)$

depends entirely on the range of f.

Recall that for an infinite cardinal r, r^+ denotes the cardinal successor of r.

<u>PROPOSITION 3.6.2</u>. Given f,g in S, $f\mathcal{L}g$ iff

(1) $f,g \in T_i$, some $i \in I$,

and

(2) either $f,g \in CT(p,p)$

or $f,g \in CT(p,r) \cap R(r^+)$ for $q \leqslant r < p$.

<u>Proof</u>. Assume firstly that $f\mathcal{L}g$. If $f \neq g$, then (3.6.2) means that there exist $s,t \in S$ with

(3.6.4) $$f = sg \quad\text{and}\quad g = tf.$$

This immediately implies that $\pi(f) \subseteq \pi(g)$ and $\pi(g) \subseteq \pi(f)$, that is $\pi(f) = \pi(g)$. Hence $f,g \in T_i$, some $i \in I$.

Since for all r, $q \leqslant r \leqslant p$, CT(p,r) is an ideal of S (Proposition 3.0.1), equations (3.6.4) imply that $f \in CT(p,r)$ iff $g \in CT(p,r)$. That means that either $f,g \in CT(p,p)$ or there exists an $r < p$ with

$$f,g \in CT(p,r) - CT(p,r^+) .$$

We show that the latter implies $f \in R(r^+)$. On the contrary, assume that $f \notin R(r^+)$. That is, for every $A \in \mathcal{E}$,

$$\text{defect}(R(f),A) \geqslant r^+ .$$

In particular, since $g = tf$, $R(g) \subseteq R(t)$ with

$$|R(t) - R(g)| = \text{defect}(R(f),\pi(t)) \geqslant r^+ .$$

Thus $g \in CT(p,r^+)$, contrary to our assumption. Hence $f \in R(r^+)$. Similarly, using the equation $f = sg$, we can show that $g \in R(r^+)$.

For the converse assume that $f,g \in T_i$, some $i \in I$. If $f,g \in CT(p,p)$ then Lemma 3.3.7(1) ensures the existence of s,t in S with

$$f = sg \quad\text{and}\quad g = tf.$$

It follows from (3.6.2) that $f\mathcal{L}g$. Hence assume that for some $r < p$

$$f,g \in CT(p,r) \cap R(r^+) .$$

We construct an s in S such that

$$f = sg.$$

Since $g \in R(r^+)$, we can choose $A \in \mathcal{E}$ with defect$(R(g),A) = r_1 \leqslant r$. Let $Y \subseteq X$ such that $Y \dot{\cup} R(g)$ is a cross-section of A. Then $|Y| = r_1$. Choose a w.s. $A \supseteq R(f)$ with $|A - R(f)| = r$, and write $A - R(f) = B \dot{\cup} C$, where $|B| = r_1, |C| = r$. Let k be a partial transformation

$$k : R(g) \to R(f),$$

given by $k(x) = fg^{-1}(x)$, $x \in R(g)$. Since $\pi(f) = \pi(g)$, k is a bijection. Choose an arbitrary bijection

$$\ell : Y \to B .$$

Define the function s on X as follows: for $x \in X$,

$$s(x) = \begin{cases} \ell(y), & \text{if } xAy, \text{ for some } y \in Y, \\ k(z), & \text{if } xAz, \text{ for some } z \in R(g). \end{cases}$$

Then $\pi(s) = A$ and $R(s) = R(f) \dot{\cup} B \subseteq A$ with $|A - R(s)| = |C| = r$, so $s \in S$ and $f = sg$.

In a similar manner we can construct $t \in S$ with

$$g = tf.$$

We deduce from (3.6.2) that $f\mathcal{L}g$. □

Recall that the equivalence $\mathcal{H}$ is the intersection of $\mathcal{L}$ and $\mathcal{R}$. By Proposition 3.6.1 $\mathcal{R} = \iota_S$, and so

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R} = \mathcal{L} \cap \iota_S = \iota_S .$$

Hence we proved

<u>PROPOSITION 3.6.3</u>. Given f,g in S,

$$f\mathcal{H}g \text{ iff } f = g.$$ □

Now, the equivalence $\mathcal{D}$ is the join of $\mathcal{L}$ and $\mathcal{R}$, where $\mathcal{R} = \iota_S$ (Proposition 3.6.1). Thus

$$\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{L} \circ \iota_S = \mathcal{L} .$$

Hence

PROPOSITION 3.6.4. Given f,g in S,

$$f\mathcal{D}g \quad \text{iff} \quad f\mathcal{L}g . \qquad \square$$

To describe the equivalence $\mathcal{J}$ we need the following definition. Given a cardinal r, $q \leqslant r \leqslant p$, let

$$E(r) = \{f \in S \mid \text{for some A in } C_q, \ \text{defect}(A,\pi(f)) \leqslant r\} .$$

Clearly, for some r, $E(r)$ may be empty. While the definition of $R(r)$ is concerned with ranges of transformations, the definition of $E(r)$ depends entirely on partitions of transformations.

PROPOSITION 3.6.5. Given f,g in S, $f\mathcal{J}g$ iff

either $f,g \in CT(p,p)$

or $f\mathcal{L}g$

or $f,g \in CT(p,r) \cap R(r^+) \cap E(r)$, for some r, $q \leqslant r < p$.

Proof. It follows from (3.6.3) that $f\mathcal{J}g$ if and only if there exist $k,\ell,s,t \in S^1$ with

$$f = kg\ell \quad \text{and} \quad g = sft . \tag{3.6.5}$$

Firstly assume that there exist $k,\ell,s,t \in S^1$ such that (3.6.5) holds. Since for all r, $q \leqslant r \leqslant p$, CT(p,r) is an ideal of S (Proposition 3.0.1), equations (3.6.5) imply that either $f,g \in CT(p,p)$, or there exists $r < p$ with

$$f,g \in CT(p,r) - CT(p,r^+) .$$

We show that the latter implies that $f \in R(r^+) \cap E(r)$. That $g \in R(r^+) \cap E(r)$ can be shown in a similar fashion.

Consider the following two cases:

Case 1. $s,t \neq \iota_X$.

If $f \notin R(r^+)$, that is for every $A \in \mathcal{E}$, $\text{defect}(R(f),A) \geqslant r^+$, then the equation $g = sft$ implies that

$$\begin{aligned} |R(s) - R(g)| &= \text{defect}(R(ft),\pi(s)) \\ &\geqslant \text{defect}(R(f),\pi(s)), \text{ since } R(ft) \subseteq R(f), \\ &\geqslant r^+ . \end{aligned}$$

Hence $g \in CT(p,r^+)$, because $R(g) \subseteq R(s)$. We conclude that $f \in R(r^+)$.

If $f \notin E(r)$, that is for every $A \in C_q$, $\text{defect}(A,\pi(f)) > r$, then $g = sft$ implies that

$$\begin{aligned} |R(sf) - R(g)| &= \text{defect}(R(t),\pi(sf)) \\ &= \text{defect}(R(t),\pi(f)), \text{ since } \pi(sf) = \pi(f), \\ &> r . \end{aligned}$$

Since $R(g) \subseteq R(sf)$, the latter means that $g \in CT(p,r^+)$. We conclude that $f \in E(r)$.

Case 2. $\iota_X \in \{s,t\}$.

If $s = t = \iota_X$, then $f = g$, and the result is trivial. Hence assume that either s or t is in S.

Let $s = \iota_X$. If in (3.6.5) $k \neq \iota_X$, then $f = kg\ell$ implies that

$$g = ft = kg\ell t = kft\ell t = kf\ell' ,$$

where $\ell' = t\ell t \in S$. Then Case 1 is applicable. Now, if $k = \iota_X$, then (3.6.5) can be rewritten as

$$f = g\ell \quad \text{and} \quad g = ft .$$

It follows from (3.6.1) that $f\mathcal{R}g$ and hence, by Proposition 3.6.1, $f = g$.

Now let $t = \iota_X$. Again, if in (3.6.5) $\ell \neq \iota_X$, then $f = kg\ell$ implies that

$$g = sf = skg\ell = sksf\ell = s'f\ell ,$$

where $s' = sks \in S$. Then Case 1 is applicable. If $\ell = \iota_X$ then (3.6.5) can be rewritten as

$$f = kg \quad \text{and} \quad g = sf .$$

We deduce from (3.6.2) that $f\mathcal{L}g$.

For the converse assume firstly that $f,g \in CT(p,p)$. Let $\ell \in S$ with $\pi(\ell) = \pi(f)$. Then $\pi(g\ell) = \pi(f)$ and Lemma 3.3.7(1) implies that there exists $k \in S$ such that

$$f = kg\ell .$$

Similarly we can show that

$$g = sft ,$$

for some $s,t \in S$. Thus $f\mathcal{J}g$.

Now assume that $f\mathcal{L}g$. Since $\mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$, we have $f\mathcal{J}g$.

Finally assume that $f,g \in CT(p,r) \cap R(r^+) \cap E(r)$, for some r, $q \leqslant r < p$. We construct $k,\ell \in S$ such that $f = kg\ell$. Let $A \in C_q$ with

(3.6.6) $$\operatorname{defect}(A,\pi(g)) \leqslant r$$

(such an A exists since $g \in E(r)$). Choose an $\ell \in S$ with

$$R(\ell) = A \quad \text{and} \quad \pi(\ell) = \pi(f) .$$

Since $g \in R(r^+)$ we can choose an $\mathcal{A}$ in $\mathcal{E}$ with

(3.6.7) $$\operatorname{defect}(R(g),\mathcal{A}) \leqslant r .$$

Then

$$\begin{aligned} \operatorname{defect}(R(g\ell),\mathcal{A}) &= \operatorname{defect}(R(g),\mathcal{A}) + |R(g) - R(g\ell)| \\ &= \operatorname{defect}(R(g),\mathcal{A}) + \operatorname{defect}(R(\ell),\pi(g)) \\ &\leqslant r, \text{ because of (3.3.6) and (3.6.7).} \end{aligned}$$

Hence $g\ell \in R(r^+)$. Since $g\ell \in CT(p,r)$, Proposition 3.6.2 implies that $f\mathcal{L}g\ell$. So there exists $k \in S^1$ such that $f = kg\ell$. In a similar manner we can construct $s,t \in S^1$ with $g = sft$. From (3.6.3) we deduce that $f\mathcal{J}g$. $\square$

<u>REMARK</u>. The results of Propositions 3.6.2 and 3.6.5 agree with the previously known facts about ideal structure of S (Section 3.0). Indeed, by Proposition 3.6.2 for each $i \in I$ the subsemigroup $T_i \cap CT(p,p)$ of S consists of a single $\mathcal{L}$ class. Thus $T_i \cap CT(p,p)$ is left simple (recall that the T_i are minimal left ideals of $CT(p,p)$). Also Proposition 3.6.5 implies that $CT(p,p)$ consists of a single $\mathcal{J}$ class, that is, $CT(p,p)$ is simple (recall that the $CT(p,p)$ is the minimal ideal of S). □

CHAPTER 4

ON THE SET OF RANGES OF A TRANSFORMATION SEMIGROUP

4.0. INTRODUCTION

Let X be an infinite set, S be a semigroup of transformations (total or partial) of X. We denote the set of ranges, or the *range family*, of S by

$$R(S) = \{R(f) \mid f \in S\} .$$

Then R(S) is a subset of the power set of X, $\mathcal{P}_X$.

There arises the problem of describing precisely those subsets of $\mathcal{P}_X$ which serve as range families of semigroups of transformations of X. The problem was suggested by B.M. Schein and to our knowledge has been solved only for the case of range families of monogenic [7, page 8] semigroups of partial transformations by P.M. Olonichev (see [17] and Section 1.2).

A subset $\mathcal{A}$ of $\mathcal{P}_X$ is said to be *normal* if for every $h \in \mathcal{G}_X$,

$$h(\mathcal{A}) = \mathcal{A} ,$$

where $h(\mathcal{A}) = \{h(A) \mid A \in \mathcal{A}\}$. Equivalently, $\mathcal{A}$ is normal if whenever $A \in \mathcal{A}$, and B is a subset of X such that $|B| = |A|$ and $|B'| = |A'|$ (where $B' = X - B$), then $B \in \mathcal{A}$. In particular, if $A \in \mathcal{A}$ and $|A| < |X|$ then $B \in \mathcal{A}$ for any subset B of X such that $|B| = |A|$.

As a step towards a solution of the above problem, we characterise all normal subsets of $\mathcal{P}_X$ which serve as range families of semigroups of *total* transformations, and, in particular, as range families of constant-free semigroups of total transformations. We also show that if a normal subset $\mathcal{A}$ of $\mathcal{P}_X$ is a range family of a semigroup of total transformations of X, then $\mathcal{A}$ is also a range family of some

G_X-normal semigroup of total transformations. In what follows by a transformation we mean a total transformation. For a transformation f of X the *defect* of f (denoted def f) is the cardinality of X - R(f).

4.1. THE RESULTS

We start with the following two lemmas. Both proofs are straightforward.

LEMMA 4.1.1. If f and g are 1-1 transformations from X into X, then:

(1) fg is a 1-1 transformation ;

(2) rank fg = rank f ;

(3) def fg = def f + def g . □

LEMMA 4.1.2. If f and g are arbitrary transformations from X into X, then

$$\text{rank } fg \leqslant \min(\text{rank } f, \text{rank } g) . \qquad \square$$

For a semigroup S of transformations of X, let

$$\sigma\text{-rank } S = \{\text{rank } f \mid f \in S\} ,$$

termed the *spectrum of ranks* of S,

$$\sigma\text{-def } S = \{\text{def } f \mid f \in S\} ,$$

termed the *spectrum of defects* of S.

By analogy, for a given $\mathcal{A} \subseteq \mathcal{P}_X$ let

$$\sigma\text{-rank } \mathcal{A} = \{|A| \mid A \in \mathcal{A}\}$$

be the *spectrum of ranks* of $\mathcal{A}$, and

$$\sigma\text{-def } \mathcal{A} = \{|A'| \mid A \in \mathcal{A}\} ,$$

be the *spectrum of defects* of $\mathcal{A}$.

Amongst all subsets of X, we distinguish sets of cardinality $|X|$ and call them *ℓ-sets* (informally called "large" sets). "Large" sets A, having

$$|A| = |A'| = |X|$$

will be called m-*sets* (or, informally, "medium" sets). Note that if a normal subset $\mathcal{A}$ of $\mathcal{P}_X$ contains an m-set then it contains all m-sets.

We denote by $\mathcal{N}_X$ the set of all normal subsets of $\mathcal{P}_X$ and by $\mathcal{R}_X$ the set of all subsets of $\mathcal{P}_X$ which serve as range families for semigroups of transformations.

DEFINITION 4.1.3. A set $\mathcal{A} \in \mathcal{N}_X$ is termed *hereditary* if for each $\alpha \in \sigma$-rank $\mathcal{A}$ with $\alpha < |X|$, σ-rank $\mathcal{A}$ includes every non-zero cardinal less than α. □

LEMMA 4.1.4. Let S be a transformation semigroup with $R(S) \in \mathcal{N}_X$. Then R(S) either comprises ℓ-sets or is hereditary.

Proof. Assume there exists $\alpha \in \sigma$-rank S with $\alpha < |X|$. Let γ be any non-zero cardinal less than α. We show

$$\gamma \in \sigma\text{-rank } S .$$

Choose an f in S with rank $f = \alpha$. Then

$$|\pi(f)| = \text{rank } f = \alpha < |X| .$$

It follows that there exists $A \in \pi(f)$ with $|A| \geqslant \alpha$ (the proof is analogous to that in Lemma 2.3.1). Choose a subset $\mathcal{B}$ of $\pi(f)$, containing A, with $|\mathcal{B}| = \gamma$. Let B be a cross-section of $\mathcal{B}$ (so that $|B| = |\mathcal{B}| = \gamma$). Let C be a subset of $A - B$ with

$$|C| = \begin{cases} \alpha , & \text{if } \alpha \geqslant \aleph_0 ; \\ \alpha - \gamma, & \text{if } \alpha < \aleph_0 . \end{cases}$$

Let $D = B \mathbin{\dot{\cup}} C$. Then $|D| = \alpha$, so that $D \in R(S)$ (because R(S) is normal and $\alpha < |X|$). Choose a $g \in S$ with $R(g) = D$. Certainly, $fg \in S$ and

$$\text{rank } fg = |fg(X)| = |f(D)| = |f(B \mathbin{\dot{\cup}} C)| = \gamma . \qquad \square$$

DEFINITION 4.1.5. A set $\mathcal{A}$ in $\mathcal{N}_X$ is called *finitely additive* if

$$\text{for every } \beta_1, \dots, \beta_n \in \sigma\text{-def } \mathcal{A}, \ \sum_{i=1}^{n} \beta_i \in \sigma\text{-def } \mathcal{A} . \qquad \square$$

Our purpose is to prove the following theorem.

THEOREM 4.1.6. Let $\mathcal{A} \in \mathcal{N}_X$. Then $\mathcal{A} \in \mathcal{R}_X$ if and only if $\mathcal{A}$

(i) either comprises ℓ-sets or is hereditary;

and

(ii) either has no m-sets or is finitely additive. □

We start by constructing certain subsemigroups of the full transformation semigroup $\mathcal{T}_X$ generated by a given set of transformations.

Let $\mathcal{A} \in \mathcal{N}_X$. Given a cardinal $\alpha \in \sigma$-rank $\mathcal{A}$, let

$$\mathcal{A}_\alpha = \{A \in \mathcal{A} \mid |A| = \alpha\} .$$

For each $\alpha \in \sigma$-rank $\mathcal{A}$, choose a partition $\mathcal{C}_\alpha$ of X into α sets of cardinality $|X|$ each. Let C_α be a cross-section of $\mathcal{C}_\alpha$ (so that $|C_\alpha| = |\mathcal{C}_\alpha| = \alpha$). For each $A \in \mathcal{A}_\alpha$ choose a bijection

$$h_A : C_\alpha \to A ,$$

and define a map

$$f_A : X \to X ,$$

having the partition $\pi(f_A) = \mathcal{C}_\alpha$ and acting on C_α as h_A. Let

(4.1.1) $$\mathcal{F}_\alpha = \{f_A \mid A \in \mathcal{A}_\alpha\} .$$

DEFINITION 4.1.7. Let $\mathcal{A} \in \mathcal{N}_X$. Let S_1 be the subsemigroup of $\mathcal{T}_X$ generated by the set $\bigcup_{\alpha \in \sigma\text{-rank } \mathcal{A}} \mathcal{F}_\alpha$:

$$S_1 = \left\langle \bigcup_{\alpha \in \sigma\text{-rank } \mathcal{A}} \mathcal{F}_\alpha \right\rangle ,$$

where the $\mathcal{F}_\alpha$ are as specified above. □

LEMMA 4.1.8. Let $\mathcal{A} \in \mathcal{N}_X$ either comprise ℓ-sets or be hereditary, and have no m-sets. Then

$$R(S_1) = \mathcal{A} .$$

Proof. It readily follows from the definition of S_1 that

$$R(S_1) \supseteq \mathcal{A} .$$

We show the reverse inclusion. Let $A \in R(S_1)$ and $f \in S_1$ be such that $R(f) = A$. Choose $f_{A_1}, \ldots, f_{A_n}$ in the generating set of S_1 with

$$f = f_{A_1} \cdot \ldots \cdot f_{A_n} .$$

Then by Lemma 4.1.2

$$|A| = \text{rank } f \leqslant \text{rank } f_{A_i} = |A_i|, \ i = 1, \ldots, n.$$

If some A_i has cardinality less than $|X|$, then since $\mathcal{A}$ is hereditary, $|A| \in \sigma\text{-rank } \mathcal{A}$ (Definition 4.1.3). The normality of $\mathcal{A}$ in conjunction with the fact that $|A| < |X|$ ensures that $A \in \mathcal{A}$. Hence we assume that

$$|A_i| = |X|, \ i = 1, \ldots, n .$$

Since $\mathcal{A}$ does not contain "medium" sets, for each i

$$|A_i'| < |X| .$$

Therefore each A_i contains a cross-section of $\mathcal{C}_{|X|}$ $(= \pi(f_{A_{i-1}}))$ $i = 2, \ldots, n$. Hence

$$\begin{aligned} A = R(f) &= f_{A_1} \cdot \ldots \cdot f_{A_{n-1}} f_{A_n}(X) \\ &= f_{A_1} \cdot \ldots \cdot f_{A_{n-1}}(A_n) \\ &= f_{A_1} \cdot \ldots \cdot f_{A_{n-1}}(X) \\ &= \ldots \ \ldots \ \ldots \\ &= f_{A_1}(X) \\ &= A_1 \in \mathcal{A} , \end{aligned}$$

as required. □

Now assume $\mathcal{A}$ in $\mathcal{N}_X$ contains "large" sets and for each $A \in \mathcal{A}_{|X|}$ choose a bijection

$$k_A : X \to A .$$

Let

$$\mathcal{K} = \{k_A \mid A \in \mathcal{A}_{|X|}\} .$$

DEFINITION 4.1.9. Let $\mathcal{A} \in \mathcal{N}_X$ with $|X| \in \sigma\text{-rank } \mathcal{A}$. Define S_2 to be the subsemigroup of $\mathcal{T}_X$ generated by the set $\bigcup_{\alpha \in \sigma\text{-rank } \mathcal{A}, \alpha \neq |X|} \mathcal{F}_\alpha \cup \mathcal{K}$:

$$S_2 = \left< \bigcup_{\alpha \in \sigma\text{-rank } \mathcal{A}, \alpha \neq |X|} \mathcal{F}_\alpha \cup \mathcal{K} \right>,$$

(where $\mathcal{F}_\alpha$ are specified in (4.1.1)). □

LEMMA 4.1.10. Let $\mathcal{A} \in \mathcal{N}_X$ either comprise ℓ-sets or be hereditary and contain m-sets and be finitely additive. Then

$$R(S_2) = \mathcal{A}.$$

Proof. Clearly,

$$R(S_2) \supseteq \mathcal{A}.$$

We show the reverse inclusion. Let $A \in R(S_2)$ and $f \in S_2$ be such that $R(f) = A$. Choose $f_1, \ldots, f_n$ in the generating set of S_2 with

$$f = f_1 \cdot \ldots \cdot f_n.$$

As was done in Lemma 4.1.8, we firstly dispose of the case when rank $f_i < |X|$, some $i = 1, \ldots, n$. Indeed, since $\mathcal{A}$ is hereditary it contains $A = R(f)$ because $|A| = \text{rank } f \leqslant \text{rank } f_i < |X|$ (Lemma 4.1.2). Hence assume

$$\text{rank } f_i = |X|, \ i = 1, \ldots, n.$$

Immediately we have that

$$f_i \in \mathcal{K}, \ i = 1, \ldots, n.$$

Lemma 4.1.1 (statements (1) and (2)) ensure that

$$|A| = \text{rank } f = \text{rank } f_1 = |X|.$$

Moreover from statement (3) of Lemma 4.1.1 we deduce that

$$|A'| = \text{def } f = \sum_{i=1}^{n} \text{def } f_i \in \sigma\text{-def } \mathcal{A},$$

because $\mathcal{A}$ is finitely additive. Now, if $|A'| < |X|$, then clearly $A \in \mathcal{A}$, while if $|A'| = |X|$, then A is a "medium" set, and so is again an element of $\mathcal{A}$. We conclude that $R(S_2) \subseteq \mathcal{A}$, and the equality follows. $\square$

We are ready to return to

Proof of Theorem 4.1.6. Assume $\mathcal{A} \in \mathcal{N}_X \cap \mathcal{R}_X$ and S is a semigroup of transformations of X such that

$$\mathcal{A} = R(S) .$$

That $\mathcal{A}$ either comprises ℓ-sets or is hereditary is the content of Lemma 4.1.4. Hence we assume that $\mathcal{A}$ contains m-sets and show that $\mathcal{A}$ is finitely additive.

Let $\beta_1, \ldots, \beta_n$ be a finite subset of σ-def $\mathcal{A}$. We show

$$\sum_{i=1}^{n} \beta_i \in \sigma\text{-def } \mathcal{A} .$$

If, for some i, $\beta_i \geq \aleph_0$, then the result is certainly true. Therefore we assume that each β_i is a finite cardinal. To construct a transformation f in S with def f $= \sum_{i=1}^{n} \beta_i$ we use induction on n. Certainly, such an f exists if n = 1. Fix n and suppose that for every subset $\beta_1, \ldots, \beta_m$ (where m < n) of finite elements of σ-def $\mathcal{A}$ there exists an f in S with

$$\text{def } f = \sum_{i=1}^{m} \beta_i , \; m < n .$$

Let $\beta_1, \ldots, \beta_n$ be a subset of n finite elements of σ-def $\mathcal{A}$. By the induction supposition there exists a g in S with

$$\text{def } g = \sum_{i=1}^{n-1} \beta_i .$$

Denote $\pi(g)$ by $\mathcal{C}$. Then because def g is finite,

$$|\mathcal{C}| = \text{rank } g = |X| .$$

Write

$$\mathcal{C} = \mathcal{C}_1 \dot{\cup} \mathcal{C}_2 ,$$

where $\mathcal{C}_1$ consists of all singletons in $\mathcal{C}$, and each element of $\mathcal{C}_2$ has cardinality no less than two. Our further proof depends upon the sizes of $\mathcal{C}_1$ and $\mathcal{C}_2$.

(i) $|\mathcal{C}_1| = |X|$.

Choose β_n elements in $\mathcal{C}_1$ and let A be the union of those elements. Then $A' \in \mathcal{A}$. Let $q \in S$ with

$$R(q) = A',$$

and $f = gq$. Then

$$\text{def } f = |X - gq(X)| = |X - g(A')| .$$

However

$$g(X) = g(A) \dot{\cup} g(A'), \quad \text{so}$$
$$X - g(A') = g(A) \dot{\cup} (X - g(X)) .$$

Hence

$$\text{def } f = |X - g(A')| = |g(A)| + |X - g(X)| = \beta_n + \sum_{i=1}^{n-1} \beta_i = \sum_{i=1}^{n} \beta_i ,$$

as required.

(ii) $|\mathcal{C}_1| < |X|$.

Observe that $|\mathcal{C}_2| = |X|$ and choose a subset $\mathcal{B}$ of $\mathcal{C}_2$ of cardinality β_n. Let B be a cross-section of $\mathcal{C} - \mathcal{B}$. We show that B is an m-set. Indeed

$$|B| = |\mathcal{C} - \mathcal{B}| = |\mathcal{C}| = |X| ,$$

while B' has non-empty intersection with each element in $\mathcal{C}_2$, and so

$$|B'| = |\mathcal{C}_2| = |X| .$$

Since $\mathcal{A}$ is normal and contains m-sets, we conclude that $B \in \mathcal{A}$. So, there exists a p in S with

$$R(p) = B .$$

Let $f = gp$, then

$$\text{def } f = |X - gp(X)| = |X - g(B)| .$$

Now

$$g(X) = g(B) \dot{\cup} g(B_1) ,$$

where B_1 is a cross section of $\mathcal{B}$, so

$$X - g(B) = g(B_1) \dot{\cup} (X - g(X)), \quad \text{or}$$

$$\operatorname{def} f = |g(B_1)| + |X - g(X)| = \beta_n + \sum_{i=1}^{n-1} \beta_i = \sum_{i=1}^{n} \beta_i ,$$

as required.

Now we assume that $\mathcal{A}$ in $\mathcal{N}_X$ satisfies conditions (i) and (ii) of the theorem. To show that $\mathcal{A} \in \mathcal{R}_X$ we construct a semigroup S of transformations of X with

$$R(S) = \mathcal{A} .$$

Given $\mathcal{A}$ as above, we consider two possible situations:

(a) $\mathcal{A}$ *satisfies* (i) *and has no* m-*sets*.

Let $S = S_1$ (Definition 4.1.7). Lemma 4.1.8 ensures that $R(S) = \mathcal{A}$.

(b) $\mathcal{A}$ *satisfies* (i), *contains* m-*sets and is finitely additive*.

Let $S = S_2$ (Definition 4.1.9). Lemma 4.1.10 ensures that $R(S) = \mathcal{A}$. □

We proceed to describe all normal subsets of $\mathcal{P}_X$ which serve as range families of *constant-free* semigroups of transformations.

<u>THEOREM 4.1.11</u>. Let $\mathcal{A} \in \mathcal{N}_X$. Then $\mathcal{A}$ is the range family of a constant-free semigroup of transformations of X if and only if $\mathcal{A}$ is in $\mathcal{R}_X$ (characterized in Theorem 4.1.6) and comprises ℓ-sets.

<u>Proof</u>. Let $\mathcal{A}$ be the range family of a semigroup S. If $\mathcal{A}$ comprises "large" sets, then clearly S is a constant-free semigroup.

For the converse assume that S is constant-free. If $\alpha \in \sigma$-rank $\mathcal{A}$ with $\alpha < |X|$, then Lemma 4.1.4 implies that $\mathcal{A}$ is hereditary.

Hence

$$1 \in \sigma\text{-rank } \mathcal{A} ,$$

and so S contains a constant transformation. We conclude that

$$\sigma\text{-rank } A = \{|X|\}\ ,$$

that is A comprises ℓ-sets. □

The theorem above illustrates the fact that the two parts of condition (i) in Theorem 4.1.6 distinguish the two types of semigroups with normal range families, namely, constant-free semigroups and semigroups containing all constant maps.

We conclude by showing that Theorems 4.1.6 and 4.1.11 give descriptions of range families of all G_X-normal and constant-free G_X-normal semigroups respectively.

Denote by RG_X the collection of range families of all G_X-normal semigroups.

<u>THEOREM 4.1.12</u>. $RG_X = N_X \cap R_X$.

<u>Proof</u>. Clearly,

$$RG_X \subseteq N_X \cap R_X\ ,$$

so, we only have to show the reverse inclusion.

Let $A \in N_X \cap R_X$ and S be a semigroup of transformations with

$$R(S) = A\ .$$

Let

$$H = \{hfh^{-1} \mid h \in G_X,\ f \in S\}\ ,$$

and S* be the semigroup generated by the set H. We show that S* is a G_X-normal semigroup and $R(S^*) = A$.

1. S* *is a* G_X*-normal semigroup.*

Let $f \in S^*$ and $h \in G_X$. We show

$$hfh^{-1} \in S^*\ .$$

Choose $f_1, f_2, \ldots, f_n$ in S and $h_1, h_2, \ldots, h_n$ in G_X such that

$$f = (h_1 f_1 h_1^{-1})(h_2 f_2 h_2^{-1}) \cdot \ldots \cdot (h_n f_n h_n^{-1}) \ .$$

Then

$$\begin{aligned} hfh^{-1} &= h(h_1 f_1 h_1^{-1})h^{-1}h(h_2 f_2 h_2^{-1})h^{-1} \cdot \ldots \cdot h(h_n f_n h_n^{-1})h^{-1} \\ &= ((hh_1)f_1(hh_1)^{-1})((hh_2)f_2(hh_2)^{-1}) \cdot \ldots \cdot ((hh_n)f_n(hh_n)^{-1}) \ . \end{aligned}$$

Since $hh_i \in \mathcal{G}_X$, $i = 1,\ldots, n$, we conclude that

$$(hh_i)f_i(hh_i)^{-1} \in \mathcal{H}, \ i = 1,\ldots, n \ .$$

Thus $hfh^{-1} \in S^*$, as required.

2. $R(S^*) = \mathcal{A}$.

From the definition of S^* it is clear that

$$R(S^*) \supseteq R(S) = \mathcal{A} \ ,$$

and we show the reverse inclusion. Let $A \in R(S^*)$. Take $f \in S^*$ with $R(f) = A$. Then there exist $h_i f_i h_i^{-1} \in \mathcal{H}$, $i = 1,\ldots, n$ such that

$$f = (h_1 f_1 h_1^{-1}) \cdot \ldots \cdot (h_n f_n h_n^{-1}) \ .$$

We show, using induction on n, that

$$A = R(f) \in \mathcal{A} \ .$$

If $n = 1$, then

$$A = R(f) = h_1 f_1 h_1^{-1}(X) = h_1 f(X) = h_1(R(f)) \in \mathcal{A} \ ,$$

because $\mathcal{A} \in \mathcal{N}_X$. Fix n and suppose that each $f \in S^*$, which is the composition of $n-1$ or fewer elements of the generating set $\mathcal{H}$, has $R(f) \in \mathcal{A}$. Let

$$f = (h_1 f_1 h_1^{-1}) \cdot \ldots \cdot (h_n f_n h_n^{-1}) \ ,$$

where $h_i f_i h_i^{-1} \in \mathcal{H}$, $i = 1,\ldots, n$. We show that $R(f) \in \mathcal{A}$. Let

$$g = (h_2 f_2 h_2^{-1}) \cdot \ldots \cdot (h_n f_n h_n^{-1}) \ ,$$

then, by the induction supposition,

$$R(g) \in \mathcal{A} \ .$$

Hence, since $\mathcal{A}$ is normal, $h_1^{-1}(R(g)) \in \mathcal{A}$, that is, there exists $p \in S$ with

$$R(p) = h_1^{-1}(R(g)) \ .$$

Now,

$$f_1 p \in S, \text{ so } R(f_1 p) \in \mathcal{A} \ .$$

Since $\mathcal{A}$ is normal,

$$A = R(f) = h(R(f_1 p)) \in \mathcal{A} \ ,$$

and so $R(S^*) \subseteq \mathcal{A}$. The equality follows.

Thus $\mathcal{A}$ is the range family of the $\mathcal{G}_X$-normal semigroup S^*, or

$$\mathcal{A} \in \mathcal{RG}_X \ .$$

Hence

$$\mathcal{N}_X \cap \mathcal{R}_X \subseteq \mathcal{RG}_X \ ,$$

and the result follows. $\square$

CHAPTER 5

ORDER-AUTOMORPHISMS OF NORMAL SUBSETS OF A POWER SET

5.0. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ be a subset of $\mathcal{P}_X$. We shall view $\mathcal{A}$ as a partially ordered set, the order being given by set inclusion, and write it as $(\mathcal{A},\subseteq)$. Recall that a bijection H of $\mathcal{A}$ is called an *order-automorphism* of $\mathcal{A}$, if H and H^{-1} preserve the natural ordering of $\mathcal{A}$, that is, for every $A,B \in \mathcal{A}$

$$A \subseteq B \text{ if and only if } H(A) \subseteq H(B) .$$

The set O-Aut $\mathcal{A}$ of all order-automorphisms of $\mathcal{A}$ forms a group with composition given by

$$H_1H_2(A) = H_1(H_2(A)) ,$$

where $H_1,H_2 \in$ O-Aut $\mathcal{A}$ and $A \in \mathcal{A}$.

We say that $H \in$ O-Aut $\mathcal{A}$ is *induced* if there exists an h in $\mathcal{G}_X$ such that for every $A \in \mathcal{A}$

$$H(A) = h(A) (= \{h(x) \mid x \in A\}) .$$

We are interested in the problem of describing the group O-Aut $\mathcal{A}$ for various $\mathcal{A} \subseteq \mathcal{P}_X$. Specifically, we are concerned with two questions:

(1) the characterization of all induced order-automorphisms of $\mathcal{A}$;

and

(2) the characterization of those $\mathcal{A}$ in $\mathcal{P}_X$, for which all order-automorphisms are induced.

Apart from being of independent interest, this problem is connected with the study of automorphisms of transformation semigroups. Namely, if an automorphism ϕ of a transformation semigroup S is inner then ϕ produces an *induced* order-automorphism of the set R(S) of ranges of all transformations in S. On the other hand, in instances where an automorphism ϕ of S yields an order-automorphism of R(S), the knowledge that all order-automorphisms of R(S) are induced, can be a first step in showing that ϕ is inner. See for example [8], [10] and Chapter 3.

We solve problems (1) and (2) for $\mathcal{A} \in \mathcal{N}_X$, the set of all normal subsets of $\mathcal{P}_X$. In Theorem 5.1.12 we establish necessary and sufficient conditions for an order-automorphism of $\mathcal{A}$ in $\mathcal{N}_X$ to be induced. Theorem 5.2.13 gives necessary and sufficient conditions for $\mathcal{A}$ in $\mathcal{N}_X$ to have only induced order-automorphisms. Moreover we show that provided $\mathcal{A} \neq \{\Phi\}$ or $\{X\}$, the group O-Aut $\mathcal{A}$ is isomorphic to $\mathcal{G}_X$ (Proposition 5.2.14).

<u>REMARK 5.0.1</u>. If $\mathcal{A} = \{\Phi\}$ or $\{X\}$, then O-Aut $\mathcal{A}$ consists of $\iota_{\mathcal{A}}$, which is induced by any bijection of X. Therefore in the sequel we assume that

$$\mathcal{A} \neq \{\Phi\} \quad \text{or} \quad \{X\} .$$

Moreover, we assume that

$$\Phi, X \notin \mathcal{A} .$$

Indeed, if $\{X\} \subsetneq \mathcal{A}$ ($\{\Phi\} \subsetneq \mathcal{A}$), then for every $H \in$ O-Aut $\mathcal{A}$, $H(X) = X$ ($H(\Phi) = \Phi$). Hence H is induced if and only if the restriction of H to the set $\mathcal{A} - \{X\}$ ($\mathcal{A} - \{\Phi\}$) is induced. □

We say that $(\mathcal{A}, \subseteq)$ is *trivial* if for every A and B in $\mathcal{A}$

$$A \subseteq B \quad \text{implies} \quad A = B .$$

<u>LEMMA 5.0.2</u>. Let $\mathcal{A} \in \mathcal{N}_X$, then the ordered pair $(\mathcal{A}, \subseteq)$ is trivial if and only if

$$\{\sigma\text{-rank } \mathcal{A},\ \sigma\text{-def } \mathcal{A}\} = \{\{n\},\ \{|X|\}\} ,$$

for some fixed $n \in \mathbb{N}$.

<u>Proof</u>. If $\{\sigma\text{-rank } \mathcal{A}, \sigma\text{-def } \mathcal{A}\} = \{\{n\}, \{|X|\}\}$, then $(\mathcal{A}, \subseteq)$ is certainly trivial.

For the converse assume $(\mathcal{A}, \subseteq)$ is trivial. Since $\mathcal{A} \in \mathcal{N}_X$ we conclude that

$$|\sigma\text{-rank } \mathcal{A}| = |\sigma\text{-def } \mathcal{A}| = 1 .$$

Let

$$\sigma\text{-rank } \mathcal{A} = \{\alpha\}, \ \sigma\text{-def } \mathcal{A} = \{\beta\} ,$$

where α and β are cardinals not exceeding $|X|$. If both α and β are infinite and $A \in \mathcal{A}$, then every proper subset B of A such that

$$|B| = \alpha \quad \text{and} \quad |A - B| \leqslant \beta$$

is an element of $\mathcal{A}$, so $(\mathcal{A}, \subseteq)$ is non-trivial, a contradiction. Hence either α or β is finite, that is equal to some $n \in \mathbb{N}$. Since $\alpha + \beta = |X|$, the desired result follows. $\square$

If $\mathcal{A}$ is a normal subset of $\mathcal{P}_X$, then for each $h \in \mathcal{G}_X$ the mapping H of $\mathcal{A}$ given by

$$H(A) = h(A), \text{ each } A \in \mathcal{A} ,$$

is an order-automorphism of $\mathcal{A}$, specifically, an induced order-automorphism of $\mathcal{A}$. We show that $\mathcal{A} \in \mathcal{N}_X$ possesses non-induced order-automorphisms if and only if $(\mathcal{A}, \subseteq)$ is trivial and $\sigma\text{-rank } \mathcal{A} \neq \{1\}$, $\sigma\text{-def } \mathcal{A} \neq \{1\}$.

5.1. NECESSARY AND SUFFICIENT CONDITIONS FOR AN ORDER-AUTOMORPHISM TO BE INDUCED

Let $\mathcal{A} \in \mathcal{N}_X$. Our main task in this section is to establish a criterion for deciding whether a given order-automorphism H of $\mathcal{A}$ is induced. We will also assemble some useful machinery for our work in the following section.

To show that an order-automorphism H of $\mathcal{P}_X$ is induced, we can use the fact that H maps singletons onto singletons, and that the restriction of H to singletons provides a bijection of X which induces H. When we consider an arbitrary normal $\mathcal{A}$, in a similar fashion we use the restriction of H $\in$ O-Aut $\mathcal{A}$ to the smallest available sets in $\mathcal{A}$ which can intersect pairwise in singletons. If such smallest sets are not available in $\mathcal{A}$, they are available in $\mathcal{A}^*$, the reflection of $\mathcal{A}$ (Definition 5.1.4). Hence the two-sided nature of Theorem 5.1.12.

DEFINITION 5.1.1. Let $\mathcal{A} \subseteq \mathcal{P}_X$, $\alpha \in \sigma$-rank $\mathcal{A}$ and $\beta \in \sigma$-def $\mathcal{A}$. Then

$$\mathcal{A}_{\alpha,\beta} = \{A \in \mathcal{A} \mid |A| = \alpha,\ |A'| = \beta\}\ ,$$
$$\mathcal{A}_{\alpha} = \bigcup_{\beta} \mathcal{A}_{\alpha,\beta} = \{A \in \mathcal{A} \mid |A| = \alpha\}\ . \qquad \square$$

REMARK 5.1.2. In future we use the following observation: if $\mathcal{A}_{\alpha,\beta} \subseteq \mathcal{A}$ with $\alpha < |X|$, then $\beta = |X|$ and so

$$\mathcal{A}_{\alpha,\beta} = \mathcal{A}_{\alpha,|X|} = \mathcal{A}_{\alpha}\ . \qquad \square$$

We show that for every $\mathcal{A} \in \mathcal{N}_X$ and H $\in$ O-Aut $\mathcal{A}$, H fixes each $\mathcal{A}_{\alpha,\beta}$. That is, $H(\mathcal{A}_{\alpha,\beta}) = \mathcal{A}_{\alpha,\beta}$. We start by presenting the following definition.

DEFINITION 5.1.3. Let $\mathcal{A} \in \mathcal{N}_X$. Then

$$\mathcal{A}_S = \bigcup_{\alpha \in \sigma\text{-rank}\,\mathcal{A}} \mathcal{A}_{\alpha,|X|}\ ,$$

informally called the set of "small" sets in $\mathcal{A}$, and

$$A_L = \bigcup_{\beta \in \sigma\text{-def } A} A_{|X|,\beta}\ ,$$

the set of "large" sets, or ℓ-sets, in A. □

Note that if $A_L \cap A_S$ is non-empty, then it is equal to $A_{|X|,|X|}$, the set of m-sets. It is clear, that both A_S and A_L are normal, provided A is normal.

<u>DEFINITION 5.1.4</u>. Let $A \in N_X$, then

$$A^* = \{A' \mid A \in A\}$$

is termed the *reflection* of A . □

Observe that $A^* \in N_X$ and σ-rank $A^* = \sigma$-def A, σ-def $A^* = \sigma$-rank A .

<u>DEFINITION 5.1.5</u>. The map

$$K : A \to A^*\ ,$$

given by $K(A) = A'$, for each $A \in A$, is termed the *reflection map*. Certainly, K is a bijection. Moreover, K is an *anti-order-automorphism*, that is, for every A and B in A,

$$A \subseteq B \text{ if and only if } K(A) \supseteq K(B)\ . \qquad □$$

<u>LEMMA 5.1.6</u>. For every $H \in$ O-Aut A, let

$$\Lambda(H) = KHK^{-1}\ .$$

Then Λ is an isomorphism from O-Aut A onto O-Aut A^*, and so

$$\text{O-Aut } A \cong \text{O-Aut } A^*\ .$$

<u>Proof</u>. Firstly we observe that

$$\Lambda(H) \in \text{O-Aut } A^*\ ,$$

for each $H \in$ O-Aut A. Indeed, since H and K are bijections, $\Lambda(H)$ is a bijection of A^*. Moreover, since K and K^{-1} are anti-order-automorphisms (Definition 5.1.5), $\Lambda(H)$ and $\Lambda(H)^{-1}$ preserve the natural ordering of A^*.

In a parallel way we define the map

$$\Delta : \text{O-Aut } \mathcal{A}^* \to \text{O-Aut } \mathcal{A},$$

given by $\Delta(G) = K^{-1}GK$, for each $G \in$ O-Aut $\mathcal{A}^*$. Certainly, Δ is the inverse of Λ, and so Λ is a bijection.

Clearly, Λ is a homomorphism, and hence it is an isomorphism. □

PROPOSITION 5.1.7. Let $\mathcal{A} \in N_X$, $H \in$ O-Aut $\mathcal{A}$, then

$$H(\mathcal{A}_{\alpha,\beta}) = \mathcal{A}_{\alpha,\beta},$$

for every $\alpha \in \sigma$-rank $\mathcal{A}$, $\beta \in \sigma$-def $\mathcal{A}$.

Proof. For the purpose of our proof it is convenient to introduce the following notation:

$$\mathcal{B}_\alpha = \{A \in \mathcal{A} \mid |A| \geq \alpha\} = \mathcal{A} - \bigcup_{\delta<\alpha} \mathcal{A}_\delta, \tag{5.1.1}$$

where $\alpha \in \sigma$-rank $\mathcal{A}$.

Our proof goes via the following steps:

Step 1. *For every finite* $\alpha \in \sigma$-rank $\mathcal{A}$,

$$A \in \mathcal{A}_\alpha \textit{ iff } A \in \mathcal{B}_\alpha \textit{ and } \quad \forall B \in \mathcal{B}_\alpha,\ B \subseteq A \textit{ implies } B = A.$$

If $A \in \mathcal{A}_\alpha$, then certainly $A \in \mathcal{B}_\alpha$. Also, for every $B \in \mathcal{B}_\alpha$, $B \subseteq A$,

$$\alpha = |A| \geq |B| \geq \alpha,$$

which implies $|B| = \alpha$, and so $B = A$, since α is finite.

For the converse, $A \in \mathcal{B}_\alpha$ implies $|A| \geq \alpha$. Suppose $|A| > \alpha$, then any proper subset B of A of cardinality α is an element of $\mathcal{B}_\alpha$ (since $\alpha \in \sigma$-rank $\mathcal{A}$), with $B \subsetneq A$. Hence $|A| = \alpha$.

Step 2. *If* $\alpha \leq \beta$ *are infinite cardinals in* σ-rank $\mathcal{A}$ *and* $A \in \mathcal{A}_\beta$, *then there exists a well-ordered chain of* β *distinct subsets of* A *which are elements of* $\mathcal{B}_\alpha$. SEE ERRATA

Partition A into β disjoint subsets A_j of cardinality α each, where $j \in J$ and J is a well-ordered index set, $|J| = \beta$. Then $\mathcal{D} = \{\bigcup_{i\leq j} A_i\}_{j\in J}$ is the required chain.

Now, for an infinite cardinal α in σ-rank $\mathcal{A}$ and an A in $\mathcal{A}$, let $\mathcal{C}_\alpha(A)$ be the set of all well-ordered chains of distinct subsets of A which are elements of $\mathcal{B}_\alpha$. Clearly, if $|A| < \alpha$, then $\mathcal{C}_\alpha(A) = \Phi$, while if $|A| \geqslant \alpha$, Step 2 ensures that $\mathcal{C}_\alpha(A)$ is non-empty.

Step 3. *For every infinite* $\alpha \in \sigma$-rank $\mathcal{A}$ *and* $A \in \mathcal{A}$,

$$A \in \mathcal{A}_\alpha \textit{ iff } \max_{\mathcal{D} \in \mathcal{C}_\alpha(A)} |\mathcal{D}| = \alpha .$$

Assume $A \in \mathcal{A}_\alpha$. Step 2 ensures that there exists a $\mathcal{D} \in \mathcal{C}_\alpha(A)$ with $|\mathcal{D}| = \alpha$. For any other $\mathcal{D}$ in $\mathcal{C}_\alpha(A)$, if $D_j \in \mathcal{D}$, then $D_{j+1} = D_j \dot{\cup} B_j$, for some $B_j \subseteq A$ with $B_j \neq \Phi$, since $\mathcal{D}$ comprises distinct elements. Then

$$|\mathcal{D}| \leqslant |\dot{\bigcup_{j \in J}} B_j| \leqslant |A| = \alpha, \quad \text{for } |J| > 1 ,$$

because $\dot{\bigcup}_{j\in J} B_j \subseteq A$.

Now, let $A \in \mathcal{A}$. Clearly, $\max_{\mathcal{D} \in \mathcal{C}_\alpha(A)} |\mathcal{D}|$ exists and we assume that it equals α. Let $\mathcal{D} \in \mathcal{C}_\alpha(A)$ with $|\mathcal{D}| = \alpha$, then

$$\begin{aligned} |A| &\geqslant |\mathcal{D}|, \quad \text{as in the previous paragraph} \\ &= \alpha . \end{aligned}$$

If $|A| > \alpha$, Step 2 ensures that there exists $\mathcal{D} \in \mathcal{C}_\alpha(A)$ with $|\mathcal{D}| = |A| > \alpha$, a contradiction. We conclude that $|A| = \alpha$.

Step 4. *For every* $\alpha \in \sigma$-rank $\mathcal{A}$, $H(\mathcal{A}_\alpha) = \mathcal{A}_\alpha$.

The set σ-rank $\mathcal{A}$ of cardinals can be well-ordered and we write:

$$\sigma\text{-rank } \mathcal{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_j, \ldots\} ,$$

where $0 < \alpha_i < \alpha_j \leqslant |X|$ for $i < j$ (Remark 5.0.1). We use induction on the index j.

Let j = 1. Clearly $\mathcal{B}_{\alpha_1} = \mathcal{A}$. If α_1 is finite, we use Step 1 to get the required result. Assume that α_1 is infinite. Then

$$\mathcal{D} \in \mathcal{C}_{\alpha_1}(A) \text{ iff } H(\mathcal{D}) \in \mathcal{C}_{\alpha_1}(H(A)) ,$$

where $H(\mathcal{D}) = \{H(D) \mid D \in \mathcal{D}\}$. Hence we derive the result from Step 3. We conclude that

$$H(\mathcal{A}_{\alpha_1}) = \mathcal{A}_{\alpha_1} .$$

Fix j and assume the result is true for each $i < j$. Then

(5.1.2) $$H(\mathcal{B}_{\alpha_i}) = \mathcal{B}_{\alpha_i}, \text{ each } i \leqslant j \text{ (see (5.1.1))} .$$

We show that $H(\mathcal{A}_{\alpha_j}) = \mathcal{A}_{\alpha_j}$. If $\alpha_j < \aleph_0$, the result follows from Step 1 and (5.1.2), while if $\aleph_0 \leqslant \alpha_j \leqslant |X|$, Step 3 in conjunction with (5.1.2) ensure the result. The induction is complete.

It readily follows from Step 4 and Remark 5.1.2 that

(5.1.3) $$H(\mathcal{A}_{\alpha,|X|}) = \mathcal{A}_{\alpha,|X|} ,$$

SEE ERRATA

for all $\alpha \in \sigma\text{-rank } \mathcal{A}$. If $\sigma\text{-def } \mathcal{A} = \{|X|\}$, the proof is complete. Hence assume that $\sigma\text{-def } \mathcal{A} \neq \{|X|\}$.

Step 5. *For every* $\beta \in \sigma\text{-def } \mathcal{A}$, $H(\mathcal{A}_{|X|,\beta}) = \mathcal{A}_{|X|,\beta}$.

Let $\mathcal{A}^*$ be the reflection of $\mathcal{A}$ (Definition 5.1.4). Recall that $\sigma\text{-rank } \mathcal{A}^* = \sigma\text{-def } \mathcal{A}$, and for each $\beta \in \sigma\text{-def } \mathcal{A}$ let

$$\mathcal{A}^*_\beta = \{A \in \mathcal{A}^* \mid |A| = \beta\} .$$

Let K be the reflection map (Definition 5.1.5). Then $KHK^{-1} \in \text{O-Aut } \mathcal{A}^*$ (Lemma 5.1.6), and for each $\beta \in \sigma\text{-def } \mathcal{A}$ such that $\beta < |X|$,

$$\begin{aligned} H(\mathcal{A}_{|X|,\beta}) &= K^{-1}KHK^{-1}(\mathcal{A}^*_\beta) = K^{-1}(\mathcal{A}^*_\beta), \quad \text{by Step 4} \\ &= \mathcal{A}_{|X|,\beta} . \end{aligned}$$

Finally, if $|X| \in \sigma\text{-rank } \mathcal{A} \cap \sigma\text{-def } \mathcal{A}$, then the equality

$$H(\mathcal{A}_{|X|,|X|}) = \mathcal{A}_{|X|,|X|}$$

follows by elimination.

Step 5 in conjunction with (5.1.3) produces the desired result. □

Our next theorem (Theorem 5.1.10), while being formulated for a normal set containing "small" sets, will give us (with the aid of Lemma 5.1.6) a means of deciding whether a given order-automorphism of an arbitrary normal set $\mathcal{A}$ is induced.

NOTATION 5.1.8. Given $A \in N_X$, let $\mu = \min\{\alpha \mid \alpha \in \sigma\text{-rank } A\}$ (such μ exists by Zorn's Lemma). □

We will use the following lemma. Let Y be a non-empty subset of X of cardinality α. Given $n \in \mathbb{N}$, let $P_n(Y)$ be the collection of all distinct n-element subsets of Y.

LEMMA 5.1.9. $$|P_n(Y)| = \begin{cases} \binom{\alpha}{n} , & \alpha < \aleph_0 ; \\ \alpha , & \alpha \geqslant \aleph_0 . \end{cases}$$

The proof is straightforward. □

Now we are ready to present

THEOREM 5.1.10. Let $A \in N_X$ be such that $A_S \neq \Phi$. Let $H \in$ O-Aut A. Then H is induced if and only if for every $A,B \in A_{\mu,|X|}$

(5.1.4) $$|A \cap B| = 1 \text{ iff } |H(A) \cap H(B)| = 1 .$$

Proof. If H is induced, then (5.1.4) is certainly true.

For the converse we assume that (5.1.4) holds and show that H is induced. This is done in the following three steps:

Step 1. *Given* $x \in X$, *there exists a* $y \in X$ *such that for every* $A,B \in A_{\mu,|X|}$ *with* $A \cap B = \{x\}$ *we have* $H(A) \cap H(B) = \{y\}$.

Our proof depends on the value of μ.

(i) $\mu = |X|$.

Take A and B in $A_{|X|,|X|}$ as above and let $C,D \in A_{|X|,|X|}$ such that

$$C \cap D = \{x\} .$$

We show that

$$H(C) \cap H(D) = \{y\} .$$

We can assume that

(5.1.5) $$A \cup B = X = C \cup D .$$

(If $A \cup B \neq X$ we substitute $A' \cup \{x\}$ for B. Then

$$H(A) \cap H(B) \subseteq H(A) \cap H(A' \cup \{x\}) ,$$

and, due to our assumption (5.1.4),

$$|H(A) \cap H(B)| = 1 = |H(A) \cap H(A' \cup \{x\})| .$$

Hence

$$H(A) \cap H(B) = H(A) \cap H(A' \cup \{x\}) .$$

Similar arguments apply to C and D.)

Assumption (5.1.5) allows us in turn to assume that $|A \cap C| = |X|$, that is

$$A \cap C \in A_{|X|,|X|} .$$

Also,

$$B \cup D = (A \cap C)' \cup \{x\} \in A_{|X|,|X|} .$$

Now, each of the following three pairs of sets in $A_{|X|,|X|}$:

$A \cap C$ and B; $A \cap C$ and D and $A \cap C$ and $B \cup D$

intersect precisely in the set $\{x\}$. Due to the assumption (5.1.4), we conclude that each of the inclusions below is, in fact, an equality:

$$\begin{aligned}\{y\} = H(A) \cap H(B) &\supseteq H(A \cap C) \cap H(B)\\ &\subseteq H(A \cap C) \cap H(B \cup D)\\ &\supseteq H(A \cap C) \cap H(D)\\ &\subseteq H(C) \cap H(D) .\end{aligned}$$

Hence $H(C) \cap H(D) = \{y\}$, as required.

(ii) $1 < \mu < |X|$.

We write $A_{\mu,|X|}$ as A_μ (Remark 5.1.2). Let A and B in A_μ be such that

$$A \cap B = \{x\}, \quad H(A) \cap H(B) = \{y\} .$$

Take any other pair C and D in A_μ with

$$|C \cap D| = 1 .$$

Let $\mathcal{F}$ be a subset of $\mathcal{A}_\mu$ with the properties:

(a) for every distinct F_1, F_2 in $\mathcal{F}$, $|F_1 \cap F_2| = 1$;

(b) for every F in $\mathcal{F}$, $|A \cap F| = |B \cap F| = |C \cap F| = |D \cap F| = 1$.

We show

$$C \cap D = \{x\} \text{ iff } \exists \mathcal{F} \text{ (as described above) with } |\mathcal{F}| = |X|.$$

Let $A \cup B \cup C \cup D = Y$, then $|Y| \leqslant 4\mu < |X|$ and $|Y'| = |X|$.

If $C \cap D = \{x\}$, let π be a partition of Y' into $|X|$ sets of cardinality $\mu - 1$ each. (Note: by $\mu - 1$ we mean μ, if μ is infinite, and the predecessor of μ, when μ is finite). Let

$$\mathcal{F} = \{P \cup \{x\} \mid P \in \pi\}.$$

Then $\mathcal{F}$ satisfies (a) and (b) and $|\mathcal{F}| = |\pi| = |X|$.

For the converse assume $C \cap D = \{z\}$, $z \neq x$, and $\mathcal{F} \subseteq \mathcal{A}_\mu$ satisfies (a) and (b). For each F in $\mathcal{F}$ we have $|Y \cap F| > 1$. (If not, then

$$\begin{aligned} |Y \cap F| &= |(A \cup B \cup C \cup D) \cap F| \\ &= |(A \cap F) \cup (B \cap F) \cup (C \cap F) \cup (D \cap F)| \leqslant 1. \end{aligned}$$

Using condition (b) we conclude:

$$A \cap F = B \cap F = C \cap F = D \cap F \subseteq A \cap B = \{x\},$$

or $\qquad C \cap D \supseteq \{x\}$, a contradiction.)

Also, condition (b) implies for each $F \in \mathcal{F}$

$$|Y \cap F| = |(A \cap F) \cup (B \cap F) \cup (C \cap F) \cup (D \cap F)| \leqslant 4.$$

Hence for each $F \in \mathcal{F}$,

$$1 < |Y \cap F| \leqslant 4.$$

Define a map

$$\lambda : \mathcal{F} \to \bigcup_{i=2}^{4} \mathcal{P}_i(Y)$$

via $\lambda(F) = Y \cap F$, each $F \in \mathcal{F}$. We show that λ is 1-1. Indeed, assume F_1 and F_2 in $\mathcal{F}$ are such that

$$\lambda(F_1) = \lambda(F_2) .$$

Then

$$1 < |Y \cap F_1| = |Y \cap F_1 \cap F_2| \leqslant |F_1 \cap F_2| ,$$

so that $|F_1 \cap F_2| > 1$ and condition (a) ensures

$$F_1 = F_2 .$$

Hence

$$|\mathcal{F}| \leqslant |\bigcup_{i=2}^{4} \mathcal{P}_i(Y)| < |X| \quad (|Y| < |X| \text{ and Lemma 5.1.9}).$$

This confirms that $C \cap D = \{x\}$.

Observe now that the definition of the set $\mathcal{F}$ depends on the sets A,B,C and D. We denote this dependence by writing

$$\mathcal{F} = \mathcal{F}(A,B,C,D) .$$

Hence

$$\begin{aligned} C \cap D = \{x\} \quad &\text{iff} \quad \exists \mathcal{F}(A,B,C,D) \text{ with } |\mathcal{F}(A,B,C,D)| = |X| \\ &\text{iff} \quad \exists \mathcal{F}(H(A),H(B),H(C),H(D)) \text{ with} \\ &\qquad |\mathcal{F}(H(A),H(B),H(C),H(D))| = |X| \quad \text{(Assumption 5.1.4))} \\ &\text{iff} \quad H(C) \cap H(D) = \{y\} . \end{aligned}$$

(iii) $\mu = 1$.

The result follows immediately from Proposition 5.1.7.

Now we are in a position to define a map

$$h : X \to X \text{ via } \{h(x)\} = H(A) \cap H(B) ,$$

where $A,B \in \mathcal{A}_{\mu,|X|}$ with $A \cap B = \{x\}$. (Note: if $\mu = 1$, then h coincides with the action of H on $\mathcal{A}_1$).

Step 2. h *is a bijection of* X.

That h is well-defined is the content of Step 1. By considering the order-automorphism H^{-1}, we can define a map

$$k : X \to X \text{ via } \{k(x)\} = H^{-1}(A) \cap H^{-1}(B) ,$$

where $A, B \in A_{\mu,|X|}$ with $A \cap B = \{x\}$. It is straightforward to show that k is the inverse of h, and so h is a bijection of X.

Step 3. H *is induced by* h.

Firstly we show that

$$H(A) = h(A), \text{ for each } A \in A_{\mu,|X|}.$$

From the definition of h we at once have that $h(A) \subseteq H(A)$. Take $y \in H(A)$ and let $H(B) \in A_{\mu,|X|}$ with $H(A) \cap H(B) = \{y\}$. Then there exists $x \in A$ such that $A \cap B = \{x\}$, and by the definition of h, $h(x) = y$. Hence $H(A) \subseteq h(A)$, and the equality follows.

Now let C be an arbitrary set in A. Since A is normal, the first part of the proof in conjunction with Proposition 5.1.7 implies that

$$H(C) = \bigcup_{\substack{H(A) \subseteq H(C) \\ H(A) \in A_{\mu,|X|}}} H(A) = \bigcup_{\substack{A \subseteq C \\ A \in A_{\mu,|X|}}} h(A) = h\Big(\bigcup_{\substack{A \subseteq C \\ A \in A_{\mu,|X|}}} A\Big) = h(C) \qquad \square$$

Now we generalize Theorem 5.1.10 to an arbitrary normal set A (recall, $\{\Phi\}, \{X\} \not\subseteq A$, by Remark 5.0.1.)

Let $A \in N_X$ and A^* be the reflection of A (Definition 5.1.4). Then for every $H \in$ O-Aut A there exists a unique $\Lambda(H) \in$ O-Aut A^* given by $\Lambda(H) = KHK^{-1}$ (Lemma 5.1.6). In view of Definition 5.1.5 the following result is straightforward.

<u>LEMMA 5.1.11</u>. Let $A \in N_X$ and $H \in$ O-Aut A. Then H is induced if and only if $\Lambda(H)$ is induced. $\square$

Let $\nu = \min\{\beta \mid \beta \in \sigma\text{-def } A\}$. (Note that $\nu > 0$ because of Remark 5.0.1.)

<u>THEOREM 5.1.12</u>. Let $A \in N_X$ and $H \in$ O-Aut A. Then H is induced if and only if either

(i) $|A \cap B| = 1$ iff $|H(A) \cap H(B)| = 1$, for every A and B in $\mathcal{A}_{\mu,|X|}$, when $\mathcal{A}_S \neq \Phi$; or

(ii) $|A' \cap B'| = 1$ iff $|H(A)' \cap H(B)'| = 1$,

for every A and B in $\mathcal{A}_{|X|,\nu}$, when $\mathcal{A}_L \neq \Phi$.

If both $\mathcal{A}_S$ and $\mathcal{A}_L$ are non-empty, then conditions (i) and (ii) are equivalent.

Proof. Firstly assume that $\mathcal{A}_S \neq \Phi$. Theorem 5.1.10 ensures that H is induced if and only if condition (i) holds.

Now assume that $\mathcal{A}_L \neq \Phi$. In view of Lemma 5.1.11 it suffices to show that the order-automorphism $\Lambda(H)$ of $\mathcal{A}^*$ is induced if and only if (ii) holds. Now, the reflection $\mathcal{A}^*$ contains "small" sets (because $\mathcal{A}_L \neq \Phi$). Hence Theorem 5.1.10 provides a criterion for deciding whether $\Lambda(H)$ is induced. Note that

$$\min\{\alpha \mid \alpha \in \sigma\text{-rank } \mathcal{A}^*\} = \nu (= \min \{\beta \mid \beta \in \sigma\text{-def } \mathcal{A}\}) ,$$

and denote, as usual, the set $\{A' \in \mathcal{A}^* \mid |A| = |X|, |A'| = \nu\}$ by $\mathcal{A}^*_{\nu,|X|}$. Clearly,

$$A' \in \mathcal{A}^*_{\nu,|X|} \quad \text{iff} \quad A \in \mathcal{A}_{|X|,\nu} .$$

Using Theorem 5.1.10 we can say that $\Lambda(H)$ is induced if and only if

$$|A' \cap B'| = 1 \quad \text{iff} \quad |\Lambda(H)(A') \cap \Lambda(H)(B')| = 1 ,$$

for all $A', B' \in \mathcal{A}^*_{\nu,|X|}$, or, equally, for all $A, B \in \mathcal{A}_{|X|,\nu}$. Since for any $A \in \mathcal{A}$

$$\Lambda(H)(A') = KHK^{-1}(A') = H(A)' ,$$

the above condition for $\Lambda(H)$ to be induced, is equivalent to (ii), as required.

Finally, we observe that if $\mathcal{A}_S$ and $\mathcal{A}_L$ are both non-empty, then

(i) holds iff H is induced iff (ii) holds,

that is, conditions (i) and (ii) are equivalent. □

5.3. NECESSARY AND SUFFICIENT CONDITIONS (ON $\mathcal{A} \in \mathcal{N}_X$) FOR ALL ORDER-AUTOMORPHISMS TO BE INDUCED

As in the previous section we firstly establish necessary and sufficient conditions on $\mathcal{A} \in \mathcal{N}_X$ with $\mathcal{A}_S \neq \Phi$ to have only induced order-automorphisms. Specifically, we show that all order-automorphisms of $\mathcal{A}$ (with $\mathcal{A}_S \neq \Phi$) are induced if and only if either $(\mathcal{A}, \subseteq)$ is non-trivial, or $(\mathcal{A}, \subseteq)$ is trivial and σ-rank $\mathcal{A} = \{1\}$.

REMARK 5.2.1. If $\mathcal{A} \in \mathcal{N}_X$ with $\mathcal{A}_S \neq \Phi$, then Lemma 5.0.2 implies that $(\mathcal{A}, \subseteq)$ is trivial if and only if

$$\sigma\text{-rank } \mathcal{A} = \{n\}, \text{ some } n \in \mathbb{N} .$$

Hence, $(\mathcal{A}, \subseteq)$ is non-trivial if and only if either

(i) $\mu \geqslant \aleph_0$

or

(ii) $\mu < \aleph_0$ and $\exists \alpha \in \sigma\text{-rank } \mathcal{A}, \alpha > \mu$,

where μ is the minimal element of σ-rank $\mathcal{A}$ (Notation 5.1.8). □

Our approach to the study of order-automorphisms of $\mathcal{A}$ depends on the value of μ and (in the case when μ is finite) on the value of α . We point out that the case when μ and α are both finite, proves to be the most complicated.

The next lemma displays some properties of order-automorphisms of normal sets.

LEMMA 5.2.2. Let $\mathcal{A} \in \mathcal{N}_X$, A,B,C,D $\in \mathcal{A}$ and H $\in$ O-Aut $\mathcal{A}$. Then

(1) $A \cup B \subseteq C$ iff $H(A) \cup H(B) \subseteq H(C)$;

(2) If $A \cup B$ is a subset of some member of $\mathcal{A}$, then

$D \subseteq A \cup B$ iff $H(D) \subseteq H(A) \cup H(B)$;

(3) If $A \cup B \in \mathcal{A}$, then $H(A \cup B) = H(A) \cup H(B)$;

(4) If A,B and $A \cap B \in \mathcal{A}_{\alpha,\beta}$, with $\alpha,\beta \geqslant \aleph_0$, then

$H(A \cap B) = H(A) \cap H(B)$.

Proof. (1) This follows from the observation that

$$A \cup B \subseteq C \text{ iff } A \subseteq C \text{ and } B \subseteq C .$$

(2) Since $X \notin \mathcal{A}$ (Remark 5.0.1), there exists C in $\mathcal{A}$ with $C \supseteq A \cup B$ and $C \neq X$. Hence

$$\begin{aligned} D \subseteq A \cup B \quad &\text{iff} \quad \forall C \in \mathcal{A},\ C \supseteq A \cup B \text{ implies } C \supseteq D \\ &\text{iff} \quad \forall H(C) \in \mathcal{A},\ H(C) \supseteq H(A) \cup H(B) \text{ implies } H(C) \supseteq H(D) \\ &\qquad\qquad \text{(because of (1))} \\ &\text{iff} \quad H(D) \subseteq H(A) \cup H(B). \end{aligned}$$

(3) This equality readily follows from (1) and (2).

(4) Clearly,

$$H(A \cap B) \subseteq H(A) \cap H(B) . \tag{5.2.1}$$

To show that the reverse inclusion holds, note that

$$H(A) \cap H(B) \in \mathcal{A}_{\alpha,\beta} ,$$

because

$$H(A \cap B) \subseteq H(A) \cap H(B) \subseteq H(A) ,$$

and $H(A \cap B)$, $H(A) \in \mathcal{A}_{\alpha,\beta}$ (Proposition 5.1.7), where $\alpha,\beta \geqslant \aleph_0$. By replacing A,B and H in (5.2.1) with $H(A)$, $H(B)$ and H^{-1} respectively, we get

$$H^{-1}(H(A) \cap H(B)) \subseteq H^{-1}H(A) \cap H^{-1}H(B), \quad \text{or}$$
$$H(A) \cap H(B) \subseteq H(A \cap B) ,$$

and the equality follows. □

PROPOSITION 5.2.3. Let $\mathcal{A} \in \mathcal{N}_X$, $\mathcal{A}_S \neq \Phi$ and $\mu \geqslant \aleph_0$. Then every $H \in$ O-Aut $\mathcal{A}$ is induced.

Proof. In view of Theorem 5.1.10 it is sufficient to show that given H in O-Aut $\mathcal{A}$,

$$|A \cap B| = 1 \quad \text{iff} \quad |H(A) \cap H(B)| = 1$$

for any $A, B \in \mathcal{A}_{\mu, |X|}$.

Take such an H and let $A, B \in \mathcal{A}_{\mu, |X|}$. Two cases can occur:

(i) $A \cup B \in \mathcal{A}_{\mu, |X|}$.

In order to establish a condition on A and B, which is equivalent to $|A \cap B| = 1$ and is preserved under the order-automorphism H, we show:

$$|A \cap B| = 1 \quad \text{iff} \quad \exists \text{ unique } C \in \mathcal{A} \text{ with } C \subsetneq A \subseteq B \cup C .$$

For any distinct A, B and C in $\mathcal{A}$,

$$\begin{aligned} C \subsetneq A \subseteq B \cup C \quad &\text{iff} \quad A - B \subseteq C \subsetneq A \\ &\text{iff} \quad C = (A-B) \mathbin{\dot{\cup}} D,\ D \subsetneq A \cap B. \end{aligned}$$

Hence for $C \in \mathcal{A}$, with $C \subsetneq A \subseteq B \cup C$ to be unique, D can only be the empty set, that is $|A \cap B| = 1$.

Now since $C \subsetneq A$, we have that $B \subseteq B \cup C \subseteq B \cup A$. Since B, $B \cup A \in \mathcal{A}_{\mu, |X|}$, $B \cup C \in \mathcal{A}_{\mu, |X|}$ also. With the aid of Lemma 5.2.2(3) we conclude now

$$\begin{aligned} |A \cap B| = 1 \quad &\text{iff} \quad \exists \text{ unique } H(C) \in \mathcal{A}, \text{ with } H(C) \subsetneq H(A) \subseteq H(B) \cup H(C) \\ &\text{iff} \quad |H(A) \cap H(B)| = 1. \end{aligned}$$

(ii) $A \cup B \notin \mathcal{A}_{\mu, |X|}$.

We show how this case can be reduced to (i). Since $|A \cup B| = \mu$ always (μ is infinite), we must have that $|(A \cup B)'| < |X|$, and hence

$$\mu = |X| .$$

Then

$$|A - B| = |B'| = |X| ,$$

because $B \in \mathcal{A}_{|X|, |X|}$ and $|(A \cup B)'| < |X|$. Write $A - B = E \mathbin{\dot{\cup}} F$, where $E, F \in \mathcal{A}_{|X|, |X|}$ and let $P = A - E$. Certainly $P \in \mathcal{A}_{|X|, |X|}$, and

$$B \cup P = B \cup (A-E) = (A \cup B) - E \in \mathcal{A}_{|X|, |X|} .$$

Observe now that we can substitute P for A, indeed:

$$B \cap P = B \cap (A - E) = A \cap B \qquad (E \subseteq A - B)\ ,$$

and, with the aid of Lemma 5.2.2(4) we deduce that

$$H(B) \cap H(P) = H(B) \cap H(A) \cap H(E') = H(A) \cap H(B) \quad \text{(because } B \subseteq E'\text{)}.$$

Since $B \cup P \in A_{|X|,|X|}$, the required result follows from (i). □

We continue to study order-automorphisms of a normal set A with $A_S \neq \Phi$ and $(A, \subseteq)$ being non-trivial. That means (Remark 5.2.1) that either μ is infinite (and Proposition 5.2.3 covers this possibility) or μ is finite and there exists

$$\alpha \in \sigma\text{-rank } A, \quad \alpha \neq \mu\ .$$

Our next approach to the study of order-automorphisms of A depends on the value of α .

REMARK 5.2.4. In the sequel we use the fact that the function $f(x) = \binom{x}{a}$ for $x \in \mathbb{N}$ and $x \geqslant a$ is 1-1, for any $a \in \mathbb{N}$. □

PROPOSITION 5.2.5. Let $A \in N_X$, $A_S \neq \Phi$, $\mu < \aleph_0$ and there exists an $\alpha \in \sigma$-rank A such that $\alpha \geqslant 2\mu - 1$. Then every $H \in$ O-Aut A is induced.

Proof. Take an $H \in$ O-Aut A. Due to Theorem 5.1.10 it is sufficient to show that for all $A,B \in A_\mu$

$$|A \cap B| = 1 \quad \text{iff} \quad |H(A) \cap H(B)| = 1\ .$$

(Note: $A_{\mu,|X|} = A_\mu$, by Remark 5.1.2).

For A and $B \in A_\mu$, let

$$\mathcal{D}(A,B) = \{D \in A_\mu \mid D \subseteq A \cup B\}\ .$$

Then

$$|\mathcal{D}(A,B)| = \binom{|A \cup B|}{\mu}\ .$$

Hence

$$|A \cap B| = 1 \text{ iff } |\mathcal{D}(A,B)| = \binom{2\mu - 1}{\mu} \quad \text{(Remark 5.2.4)}$$

$$\text{iff } |\mathcal{D}(H(A),H(B))| = \binom{2\mu - 1}{\mu}$$

($A \cup B$ is a subset of some member of $\mathcal{A}_{\alpha,|X|}$ and Lemma 5.2.2(2))

$$\text{iff } |H(A) \cap H(B)| = 1 \quad \text{(Remark 5.2.4).} \qquad \square$$

Due to the above result we can assume now that

$$\mu < \aleph_0 \text{ and } \alpha < 2\mu - 1 .$$

We write $\mathcal{A}_{\mu,|X|} = \mathcal{A}_\mu$ and $\mathcal{A}_{\alpha,|X|} = \mathcal{A}_\alpha$ (Remark 5.1.2).

Let $\mathbf{N}_{\alpha,\mu}$ be the initial finite segment of $\mathbf{N}$ given by

$$\mathbf{N}_{\alpha,\mu} = \{n \in \mathbf{N} \mid n\mu - (n-1)\alpha \geqslant 0\} .$$

REMARK 5.2.6. Observe that

$$\mathbf{N}_{\alpha,\mu} = \left\{1,2,\ldots, \left[\frac{\alpha}{\alpha - \mu}\right]\right\} ,$$

where [ ] is the usual largest integer function. Since $\alpha < 2\mu - 1$, $\left[\frac{\alpha}{\alpha - \mu}\right] \geqslant 2$. $\square$

Define a map

$$\tau : \mathbf{N}_{\alpha,\mu} \to \mathbf{N}$$

via
$$\tau(n) = n\mu - (n-1)\alpha, \text{ each } n \in \mathbf{N}_{\alpha,\mu} .$$

REMARK 5.2.7. Observe that τ is a strictly decreasing function, and if $m,n \in \mathbf{N}_{\alpha,\mu}$, then

$$\tau(m) + \tau(n) = m\mu - (m-1)\alpha + n\mu - (n-1)\alpha$$
$$= \tau(m+n-1) + \mu ,$$

provided $m + n - 1 \in \mathbf{N}_{\alpha,\mu}$. $\square$

Our interest in the map τ lies in the fact that for any pair of sets

A and B in $\mathcal{A}_\alpha \cup \mathcal{A}_\mu$, the property of having the cardinality of their intersection equal to $\tau(n)$ $(n \in \mathbb{N}_{\alpha,\mu})$ is preserved under any order-automorphism of $\mathcal{A}$ (Corollary 5.2.9 and Lemma 5.2.10).

LEMMA 5.2.7. Let $\mathcal{A} \in \mathcal{N}_X$, $\mu < \alpha < 2\mu - 1$, and $H \in$ O-Aut $\mathcal{A}$. Then for each A and $B \in \mathcal{A}_\mu$,

(1) $|A \cap B| \geqslant \tau(n)$ iff $|H(A) \cap H(B)| \geqslant \tau(n)$,

(2) $|A \cap B| \leqslant \tau(n)$ iff $|H(A) \cap H(B)| \leqslant \tau(n)$,

for all $n \in \mathbb{N}_{\alpha,\mu}$.

Proof. We use induction on n.

Note that $\tau(1) = \mu$. For $n = 1$, (2) holds for every A and B in $\mathcal{A}_\mu$, (1) follows from the simple observation that

$$|A \cap B| \geqslant \mu \quad \text{iff} \quad A = B .$$

Our induction step requires n to be greater than 2. Hence we prove now the result for $n = 2$.

$$\begin{aligned} |A \cup B| \geqslant \tau(2) \quad &\text{iff} \quad |A \cup B| = |A| + |B| - |A \cap B| \leqslant \alpha \\ &\text{iff} \quad \exists \text{ at least one } C \in \mathcal{A}_\alpha,\ A \cup B \subseteq C \\ &\text{iff} \quad \exists \text{ at least one } H(C) \in \mathcal{A}_\alpha,\ H(A) \cup H(B) \subseteq H(C) \\ &\qquad\qquad \text{(Proposition 5.1.7 and Lemma 5.2.2(1))} \\ &\text{iff} \quad |H(A) \cup H(B)| \leqslant \alpha \\ &\text{iff} \quad |H(A) \cap H(B)| \geqslant \tau(2), \end{aligned}$$

which confirms (1). If we exchange "$\leqslant$" and "$\geqslant$" in the above inequalities and substitute the words "at least one" for "at most one" we get a proof of (2).

Now fix $n > 2$, $n \in \mathbb{N}_{\alpha,\mu}$, and assume the result is true for $n - 1 \in \mathbb{N}_{\alpha,\mu}$. We prove the result for n. Let

$$\mathcal{C} = \{C \in \mathcal{A}_\mu \mid |A \cap C| \geqslant \tau(n-1),\ |B \cap C| \geqslant \tau(2)\} .$$

We show:

$$|\mathcal{C}| = \begin{cases} 0, & \text{if } |A \cap B| < \tau(n), \\ \text{non-zero and finite}, & \text{if } |A \cap B| = \tau(n), \\ \text{infinite}, & \text{if } |A \cap B| > \tau(n). \end{cases}$$

Our proof goes via the following steps:

Step 1. *If* $|A \cap B| < \tau(n)$, *then* $|\mathcal{C}| = 0$.

Assume $\mathcal{C}$ is non-empty and $C \in \mathcal{C}$, then

$$\mu = |C| \geqslant |(A \cup B) \cap C| = |A \cap C| + |B \cap C| - |A \cap B \cap C|$$
$$\geqslant \tau(n) + \mu - |A \cap B| \quad \text{(Remark 5.2.7)}.$$

Thus $|A \cap B| \geqslant \tau(n)$, which confirms the result.

Observe that the above chain of inequalities turns into a chain of equalities provided $|A \cap B| = \tau(n)$. In particular we then have

$$|C| = |(A \cup B) \cap C|, \text{ each } C \in \mathcal{C},$$

or

$$C \subseteq A \cup B, \text{ so } |\mathcal{C}| \leqslant \binom{|A \cup B|}{\mu}.$$

Thus we have shown

Step 2. *If* $|A \cap B| = \tau(n)$, *then* $\mathcal{C}$ *is a finite set.*

We proceed with

Step 3. *If* $|A \cap B| \geqslant \tau(n)$, *then* $\mathcal{C}$ *is non-empty.*

If $|A \cap B| > \tau(n)$, *then* $\exists C \in \mathcal{C}$, $C \not\subseteq A \cup B$.

Assume $|A \cap B| = \ell \geqslant \tau(n)$ and let $A \cap B = C_1$. Our construction of a set C in $\mathcal{C}$ depends upon the value of ℓ.

If $\ell \geqslant \tau(2)$, then choose $C_2 \subseteq (A \cup B)'$, $|C_2| = \mu - \ell$ (since we can assume $A \neq B$, $\ell < \mu$, so $C_2 \neq \Phi$). Let $C = C_1 \dot{\cup} C_2$, then $C \in \mathcal{C}$ with $C \not\subseteq A \cup B$.

If $\tau(n-1) \leqslant \ell < \tau(2)$, choose $C_3 \subseteq B - A$, $|C_3| = \tau(2) - \ell$; $C_4 \subseteq (A \cup B)'$, $|C_4| = \mu - \tau(2)$ (since $\mu > \tau(i)$, $i \neq 1$, $i \in \mathbb{N}_{\alpha,\mu}$, we have $\mu - \tau(2) > 0$, or $C_4 \neq \Phi$). Let $C = C_1 \dot{\cup} C_3 \dot{\cup} C_4$. Then $C \in \mathcal{C}$ with $C \not\subseteq A \cup B$.

Finally, if $\tau(n) \leqslant \ell < \tau(n-1)$, let $C_5 \subseteq A - B$, $|C_5| = \tau(n-1) - \ell$; $C_6 \subseteq (A \cup B)'$, $|C_6| = \ell - \tau(n)$. Let $C = C_1 \dot{\cup} C_3 \dot{\cup} C_5 \dot{\cup} C_6$, then $C \in \mathcal{C}$ with $C \not\subseteq A \cup B$ for $\ell > \tau(n)$.

Step 4. *If* $|A \cap B| > \tau(n)$, *then* $\mathcal{C}$ *is infinite.*

Step 3 ensures there exists $C \in \mathcal{C}$ with $C \not\subseteq A \cup B$. Let $x \in C - (A \cup B)$. Then for any y in $(A \cup B \cup C)'$

$$D = (C - \{x\}) \cup \{y\}$$

is also an element of $\mathcal{C}$. Since $|(A \cup B \cup C)'| = |X|$ $(A,B,C \in \mathcal{A}_\mu)$ we conclude that

$$|\mathcal{C}| \geqslant |(A \cup B \cup C)'| = |X| \geqslant \aleph_0 .$$

Steps 1-4 supply us with information about the size of the set $\mathcal{C}$. We can conclude that $|\mathcal{C}|$ and $|A \cap B|$ are mutually dependent, namely,

$$|A \cap B| \geqslant \tau(n) \quad \text{iff} \quad |\mathcal{C}| > 0 ,$$
$$|A \cap B| \leqslant \tau(n) \quad \text{iff} \quad |\mathcal{C}| < \aleph_0 .$$

The definition of the set $\mathcal{C}$ depends on sets A and B, and we denote this dependence by writing $\mathcal{C} = \mathcal{C}(A,B)$. By the induction supposition

$$C \in \mathcal{C}(A,B) \quad \text{iff} \quad H(C) \in \mathcal{C}(H(A),H(B)) .$$

Hence the cardinality of $\mathcal{C}$ is preserved under the order-automorphism H, that is

$$|\mathcal{C}(A,B)| = |\mathcal{C}(H(A),H(B))| .$$

The result follows. □

From Lemma 5.2.8 we immediately deduce

COROLLARY 5.2.9. Let $\mathcal{A} \in N_X$, $\mu < \alpha < 2\mu - 1$ and $H \in$ O-Aut $\mathcal{A}$. Then for each A and B in $\mathcal{A}_\mu$,

$$|A \cap B| = \tau(n) \quad \text{iff} \quad H(A) \cap H(B) = \tau(n) ,$$

all $n \in \mathbf{N}_{\alpha,\mu}$. □

We extend this result to pairs of sets in $A_\alpha \cup A_\mu$.

<u>LEMMA 5.2.10</u>. Let $A \in N_X$, $\mu < \alpha < 2\mu - 1$, and $H \in$ O-Aut A. Then for each A and B in $A_\alpha \cup A_\mu$

$$|A \cap B| = \tau(n) \quad \text{iff} \quad |H(A) \cap H(B)| = \tau(n),$$

each $n \in \mathbb{N}_{\alpha,\mu}$.

<u>Proof</u>. Let A and B be in $A_\alpha \cup A_\mu$. Then

$$|A \cap B| = \tau(n) \quad \text{iff} \quad \max_{C \subseteq A, C \in A_\mu} \ \max_{D \subseteq B, D \in A_\mu} |C \cap D| = \tau(n)$$

$$\text{iff} \quad \max_{H(C) \subseteq H(A), H(C) \in A_\mu} \ \max_{H(D) \subseteq H(B), H(D) \in A_\mu} |H(C) \cap H(D)| = \tau(n)$$

(Proposition 5.1.7, Lemma 5.2.8 and Corollary 5.2.9)

$$\text{iff} \quad |H(A) \cap H(B)| = \tau(n). \qquad \square$$

Finally, we are ready to show that when $\mu < \alpha < 2\mu - 1$, all order-automorphisms of A are induced.

<u>PROPOSITION 5.2.11</u>. Let $A \in N_X$, $\mu < \aleph_0$ and there exists an $\alpha \in \sigma$-rank A such that $\mu < \alpha < 2\mu - 1$. Then every order-automorphism H of A is induced.

<u>Proof</u>. In view of Theorem 5.1.10 we must show that given $H \in$ O-Aut A

$$|A \cap B| = 1 \quad \text{iff} \quad |H(A) \cap H(B)| = 1$$

for all A and B in A_μ.

Let $m = \left[\frac{\alpha}{\alpha - \mu}\right]$ be the maximal element of $\mathbb{N}_{\alpha,\mu}$ and

$$n = \begin{cases} m, & \text{if } \tau(m) > 0, \\ m-1, & \text{if } \tau(m) = 0. \end{cases}$$

Recall (Remark 5.2.6) that $m \geqslant 2$, and so $n \in \mathbb{N}_{\alpha,\mu}$. Moreover, $n - 1 \in \mathbb{N}_{\alpha,\mu}$. To show this it suffices to show that $\tau(m) = 0$ implies $m - 2 > 0$ (and so $m - 2 \in \mathbb{N}_{\alpha,\mu}$). This follows since $\tau(2) = 2\mu - \alpha > 1$, because of the conditions of the proposition.

Assume $|A \cap B| = \ell \leqslant \tau(n)$ and choose

$$D \in \mathcal{A}_\alpha \text{ with } B \subseteq D \text{ and } |A \cap D| = \tau(n) \tag{5.2.2}$$

(it is possible, since $\tau(n) \geqslant \ell$). Define a set $\mathcal{C} \subseteq \mathcal{A}_\alpha$ via

$$\mathcal{C} = \{C \in \mathcal{A}_\alpha \mid A \subseteq C,\ |B \cap C| = \tau(n),\ |C \cap D| = \tau(n-1)\}.$$

We show that the cardinality of $\mathcal{C}$ does not depend on the choice of the set D (as long as (5.2.2) is satisfied), however there is a 1-1 correspondence between $|\mathcal{C}|$ and $|A \cap B|$.

We start by showing that for any D, which satisfies (5.2.2),

$$C \in \mathcal{C} \text{ iff } C = A \,\dot{\cup}\, (A' \cap B' \cap D) \,\dot{\cup}\, E,$$

where $E \subseteq A' \cap B$, $|E| = \tau(n) - \ell$. Thus the number of elements in $\mathcal{C}$ is determined by the number of subsets E of $A' \cap B$ of cardinality $\tau(n) - \ell$.

Let $C \in \mathcal{C}$, then $C = A \,\dot{\cup}\, (A' \cap C)$ (since $A \subseteq C$). Write

$$A' \cap C = (A' \cap B' \cap C) \,\dot{\cup}\, (A' \cap B \cap C),$$

and denote $A' \cap B \cap C$ by E, then

$$|E| = |B \cap C| - |A \cap B \cap C| = |B \cap C| - |A \cap B| = \tau(n) - \ell.$$

Hence

$$C = A \,\dot{\cup}\, (A' \cap B' \cap C) \,\dot{\cup}\, E,\ E \subseteq A' \cap B,\ |E| = \tau(n) - \ell,$$

and we only have to show that

$$A' \cap B' \cap C = A' \cap B' \cap D.$$

Write

$$A' \cap B' \cap C = (A' \cap B' \cap C \cap D) \,\dot{\cup}\, (A' \cap B' \cap C \cap D').$$

Since

$$\begin{aligned} |A' \cap C \cap D'| &= |C \cap D'| - |A \cap C \cap D'| \\ &= |C| - |C \cap D| - |A| + |A \cap D| \quad (\text{since } A \subseteq C) \\ &= \alpha - \tau(n-1) - \mu + \tau(n) \\ &= 0, \end{aligned}$$

we conclude that

$$A' \cap B' \cap C = A' \cap B' \cap C \cap D \subseteq A' \cap B' \cap D .$$

In turn

$$A' \cap B' \cap D = (A' \cap B' \cap C \cap D) \dot\cup (A' \cap B' \cap C' \cap D)$$

and

$$\begin{aligned} |B' \cap C' \cap D| &= |C' \cap D| - |B \cap C' \cap D| \\ &= |D| - |C \cap D| - |B| + |B \cap C| \quad \text{(since } B \subseteq D\text{)} \\ &= \alpha - \tau(n-1) - \mu + \tau(n) \\ &= 0 , \end{aligned}$$

so,

$$A' \cap B' \cap C = A' \cap B' \cap C \cap D \subseteq A' \cap B' \cap D = A' \cap B' \cap C \cap D ,$$

or $\quad A' \cap B \cap C = A' \cap B' \cap D ,$

as required.

For the converse assume

$$C = A \dot\cup (A' \cap B' \cap D) \dot\cup E ,$$

where $E \subseteq A' \cap B$, $|E| = \tau(n) - \ell$. We show that $C \in \mathcal{C}$. Indeed:

$$\begin{aligned} |C| &= |A| + |A' \cap B' \cap D| + |E| , \quad \text{where} \\ |A' \cap B' \cap D| &= |A' \cap D| - |A' \cap B \cap D| \\ &= |D| - |A \cap D| - |B| + |A \cap B| \\ &= \alpha - \tau(n) - \mu + \ell . \end{aligned}$$

Hence

$$|C| = \mu + \alpha - \tau(n) - \mu + \ell + \tau(n) - \ell = \alpha, \quad \text{or } C \in \mathcal{A}_\alpha .$$

Certainly, $A \subseteq C$, and

$$\begin{aligned} |B \cap C| &= |(A \cap B) \dot\cup (B \cap E)| \\ &= \ell + |E| \qquad \text{(because } E \subseteq B\text{)} \\ &= \tau(n) . \end{aligned}$$

Also,

$$|C \cap D| = |A \cap D| + |A' \cap B' \cap D| + |D \cap E| .$$

Now, $E \subseteq B \subseteq D$, so $|D \cap E| = |E|$, and

$$\begin{aligned}|C \cap D| &= \tau(n) + \alpha - \tau(n) - \mu + \ell + \tau(n) - \ell \\ &= \tau(n-1),\end{aligned}$$

as required.

Having established the form of each C in $\mathcal{C}$ we deduce that

$$|\mathcal{C}| = \binom{|A' \cap B|}{|E|} = \binom{\mu - \ell}{\tau(n) - \ell} = \binom{\mu - \ell}{\mu - \ell - \tau(n) + \ell} = \binom{\mu - \ell}{\mu - \tau(n)} .$$

Remark 5.2.4 ensures that

$$|A \cap B| = 1 \quad \text{iff} \quad |\mathcal{C}| = \binom{\mu - 1}{\mu - \tau(n)} .$$

Observe now that the definition of $\mathcal{C}$ depends on the sets A,B and D. We denote this dependence by writing $\mathcal{C} = \mathcal{C}(A,B,D)$. Then

$$\begin{aligned}|A \cap B| = 1 \quad &\text{iff} \quad |\mathcal{C}(A,B,D)| = \binom{\mu - 1}{\mu - \tau(n)} \\ &\text{iff} \quad |\mathcal{C}(H(A),H(B),H(D))| = \binom{\mu - 1}{\mu - \tau(n)} \\ &\qquad \text{(Lemmas 5.2.8 and 5.2.10)} \\ &\text{iff} \quad |H(A) \cap H(B)| = 1.\end{aligned}$$

□

Our next result gives necessary and sufficient conditions for a normal set $\mathcal{A}$ with $\mathcal{A}_S \neq \Phi$ to have only induced order-automorphisms.

<u>THEOREM 5.2.12</u>. Let $\mathcal{A} \in \mathcal{N}_X$ and $\mathcal{A}_S \neq \Phi$. Then all order-automorphisms of $\mathcal{A}$ are induced if and only if

either $(\mathcal{A}, \subseteq)$ is non-trivial

or $(\mathcal{A}, \subseteq)$ is trivial and σ-rank $\mathcal{A} = \{1\}$.

Proof. Assume firstly that $(\mathcal{A}, \subseteq)$ is non-trivial. Then either μ, the minimal element of σ-rank $\mathcal{A}$, is infinite, and the result is given in Proposition 5.2.3, or μ is finite and there exists $\alpha \in \sigma$-rank $\mathcal{A}$, $\alpha > \mu$ (Remark 5.2.1). If μ is finite, the result is the content of Propositions 5.2.5 and 5.2.11.

If $(\mathcal{A}, \subseteq)$ is trivial and σ-rank $\mathcal{A} = \{1\}$, then $\mathcal{A}$ consists of all singletons in $\mathcal{P}_X$, so any order-automorphism H of $\mathcal{A}$ determines a bijection of X.

For the converse assume that $(\mathcal{A}, \subseteq)$ is trivial and σ-rank $\mathcal{A} \neq \{1\}$, that is

$$\mathcal{A} = \mathcal{A}_{n,|X|} = \mathcal{A}_n, \text{ some } n \in \mathbb{N},\ n > 1 \quad \text{(Lemma 5.0.2)}.$$

Then every bijection of $\mathcal{A}$ is an order-automorphism of $\mathcal{A}$. We show there exists non-induced H $\in$ O-Aut $\mathcal{A}$. In view of Theorem 5.1.10 it is sufficient to construct an H $\in$ O-Aut $\mathcal{A}$, such that

$$|A \cap B| = 1 \text{ while } |H(A) \cap H(B)| \neq 1,$$

some $A, B \in \mathcal{A}_n$.

Choose distinct A,B and C in $\mathcal{A}$ with

$$|A \cap B| = 1 \text{ and } |B \cap C| = 0 .$$

Let H interchange A and C and be the identity otherwise. Certainly, H is a bijection of $\mathcal{A}$, so H $\in$ O-Aut $\mathcal{A}$. However

$$|H(A) \cap H(B)| = |C \cap B| = 0 \neq |A \cap B| = 1,$$

and so H is non-induced. □

We extend Theorem 5.2.12 to an arbitrary normal set $\mathcal{A}$.

THEOREM 5.2.13. Let $\mathcal{A} \in \mathcal{N}_X$. Then every H $\in$ O-Aut $\mathcal{A}$ is induced if and if

either $(\mathcal{A}, \subseteq)$ is non-trivial,

or $(\mathcal{A}, \subseteq)$ is trivial and σ-rank $\mathcal{A} = \{1\}$ or σ-def $\mathcal{A} = \{1\}$.

Proof. If $A_S \neq \Phi$, then the result is the content of Theorem 5.2.12.

Assume $A_S = \Phi$, so that $A = A_L$. Let A^* be the reflection of A (Definition 5.1.4). The set A^* is normal and contains "small" elements. Hence Theorem 5.2.12 is applicable and we infer that all order-automorphisms of A^* are induced if and only if

either $(A^*, \subseteq)$ is non-trivial,

or $(A^*, \subseteq)$ is trivial and σ-rank $A^* = \{1\}$.

However, since σ-rank $A^* = \sigma$-def A and σ-def $A^* = \sigma$-rank A, Lemma 5.0.2 implies

$$(A^*, \subseteq) \text{ is trivial iff } (A, \subseteq) \text{ is trivial.}$$

Also,

$$\sigma\text{-rank } A^* = \{1\} \text{ iff } \sigma\text{-def } A = \{1\}.$$

We conclude that all order-automorphisms of A^* are induced if and only if

either $(A, \subseteq)$ is non-trivial,

or $(A, \subseteq)$ is trivial and σ-def $A = \{1\}$.

Since O-Aut $A \cong$ O-Aut A^* (Lemma 5.1.6), we deduce with the aid of Lemma 5.1.11 that all order-automorphisms of A are induced if and only if

either $(A, \subseteq)$ is non-trivial,

or $(A, \subseteq)$ is trivial and σ-def $A = \{1\}$.

The result follows. □

Finally, as a bonus we show that, provided $A \neq \{\Phi\}$ or $\{X\}$, the group O-Aut A is isomorphic to the group G_X of all bijections of X.

PROPOSITION 5.2.14. Let $A \in N_X$, $A \neq \{\Phi\}$ or $\{X\}$. Then

$$\text{O-Aut } A \cong G_X .$$

Proof. Assume firstly, that A is such that every $H \in$ O-Aut A is induced. Then the map

$$\Omega : \text{O-Aut } \mathcal{A} \to \mathcal{G}_X ,$$

given by $\Omega(H) = h$, where $h \in \mathcal{G}_X$ induces H, each $H \in$ O-Aut $\mathcal{A}$, is an isomorphism.

Now assume $\mathcal{A}$ possesses non-induced order-automorphisms, then either $\mathcal{A} = \mathcal{A}_{n,|X|}$ or $\mathcal{A} = \mathcal{A}_{|X|,n}$, some $n \in \mathbb{N}$, $n > 1$ (Theorem 5.2.13 and Lemma 5.0.2). Clearly,

$$\text{O-Aut } \mathcal{A} = \mathcal{G}_{\mathcal{A}} ,$$

where $\mathcal{G}_{\mathcal{A}}$ is the group of all bijections of $\mathcal{A}$. Moreover, Lemma 5.1.9 ensures

$$|\mathcal{A}| = |X| .$$

So

$$\text{O-Aut } \mathcal{A} = \mathcal{G}_{\mathcal{A}} \cong \mathcal{G}_X . \qquad \square$$

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CORRECTIONS TO "Automorphisms and Range Families of Transformation Semigroup

p4. l+ Add "... but not both".

p40. 13- Replace this paragraph by

"A final consequence (Theorem 3.5.14) of the main theorem is a characterisation of the situation in which all automorphisms are range-preserving, namely if and only if W provides & cross-sections, or the subset V of W which is th union of ranges of functions $f \in S - S^2$ is reduced, or S = S We point out in the course of the Chapter that Croisot-Teissier semigroups for which W provides & cross-sections can be characterised by means of the requirement that a certain natural congruence of S be the identity, and similarly those for which V is reduced can be characterised by a natural congruence being the identity on $S - S^2$. This suggests that automorphisms of semigroups for which these congruences are the identity might be profitably studied."

p41. 11+ Replace line by

"$C_r = \{$w.s. $A \subseteq W \mid$ for some w.s. B, $A \subseteq B$ and $|B - A| =$

p59. 11+ Replace "if" by "if and only if"

13+,14+ Replace the first sentence by

"By convention, $A \lambda B$ automatically holds if there are no non-trivial ρ-classes meeting both A and B."

p65. 12+ Replace "is just A" by "is just A^c"

p73. 8+ The map $\theta : H \to \mathrm{Inn}\, S$ such that $\theta : h \to \phi_h$ is 1-1, indeed, let $h_1, h_2 \in H$ with $\theta(h_1) = \theta(h_2)$. Then for every $f \in S$, $h_1 f h_1^{-1} = h_2 f h_2^{-1}$, or $f = h_1^{-1} h_2 f (h_1^{-1} h_2)^{-1}$. Let $h_1^{-1} h_2 = h \in$ $u \in X$ and assume $h(u) = v \neq u$. If $f \in S$ such that $f(v) = v$ and $f(u) \neq u$, then $v = f(v) = hfh^{-1}(v) = hf(u)$, or $f(u) = u$, a contradiction. To construct f as used above note that $X = W$ and so $\exists g \in S$ with $v \in R(g)$, $u \notin R(g)$. Let $g(z) = v$. If $z = v$, we let $f = g$. Otherwise using transitivity of S ($X = W$) choose $t \in S$ with $t(u) = z$ and let $f = gt$.

p81. 4+ Replace the first "N_β" by "M_β"

p103. 2- Replace "W is reduced" by "the subset $V = \bigcup_{f \in S - S^2} R(f)$ of W is reduced"

1- Delete "(independent)"

p109. 11+ Delete this line and insert (continuing line 10) "there is a natural bijection $g \mapsto (g(B_j))$ from C onto $\prod_j U_j$"

6-,5-,4- Delete the last sentence

p112. 15- Replace this line by

"For a general Croisot-Teissier semigroup $S = CT(X, \mathcal{E}, p, q)$ we set

$$V = \bigcup_{f \in S - S^2} R(f).$$

Directly in terms of $X, \mathcal{E}, p, q$, if r is the minimum defect of C_q sets in partitions $B \in \mathcal{E}$, then

$$V = \bigcup_{C_q - C_r} A.$$

(See 3.5.2.) Notice that V is a ρ-closed subset of W, and $V \neq \Phi$ if and only if $S \neq S^2$. There are (rather complex) examples in which V is a proper non-empty subset of W, although it can be shown that $V = W$ if $V \neq \Phi$ and there are less than p maximal w.s. sets in W. (All the examples presented in this Chapter have in fact only a finite numbe of maximal w.s. sets.) An unexpected consequence of Theorem 3.5.1 is the following:"

p112. 11- Replace "W" by "V"

9- Replace this REMARK 3.5.15 by

"<u>REMARK 3.5.15</u>. The second and third conditions can also be characterized algebraically in terms of S. The second is equivalent to the congruence μ of 3.2.8

$$f \; \mu \; g \quad \text{iff} \quad \ell f = \ell g \quad \forall \ell \in S$$

being the identity on $S - S^2$, while the third is equivalent to the congruence σ

$$f \; \sigma \; g \quad \text{iff} \quad f\ell = g\ell \quad \forall \ell \in S \;\; \text{and} \;\; rf = sg \;\text{for some}\; r,s \in$$

being the identity on S. Notice that $\sigma = \nu \wedge \pi$ where ν is congruence mentioned in 3.3.5, and π is the congruence "ha the same partition as" (see 3.3.7). Note also $\eta = \nu \wedge \mu$.

p113. 7+ Change the second "W" to "V"

p114. 15- Replace "W is reduced (see (1))" by "V is reduced (because then by (1), $R(f) \subseteq R(g)$ if and only if either (i) the conditions in 3.5.5(1) hold or (ii) $g \notin S^2$, and

$$rg = sg \Rightarrow rf = sf \quad \forall r,s \in S)"$$

p114. 4- Replace "(Remark 3.5.15)" by "(see Remark 3.5.15 or note by the proof of Theorem 3.5.14 that this property is equivaler to the congruence η being the identity on $S - S^2$)"

p116. 11+ Replace "W" by "V"

4- Insert a new line after this

" $V = W$"

3- Replace "W" by "V"

p118. 6–,7– Delete the last sentence

In general $\mathcal{H}$ is not a congruence: the fact that $\mathcal{L}$ and $\mathcal{R}$ are right and left congruences does not necessarily imply that $\mathcal{H} = \mathcal{L} \cap R$ is a right and left congruence. For example in $\mathcal{T}_X$,

$$f\mathcal{H}g \Leftrightarrow R(f) = R(g) \text{ and } \pi(f) = \pi(g) .$$

We show that $\mathcal{H}$ is not a right congruence. Choose a non-constant $f \in \mathcal{T}_X$ and $x,y \in R(f)$. Then $(x,y)f\mathcal{H}f$. Let $g \in \mathcal{T}_X$ with $\{f^{-1}(x)\} \subsetneq R(g)$, $\{f^{-1}(y)\} \cap R(g) = \Phi$. Then $x \in R(fg) - R((x,y)fg)$, so fg and $(x,y)fg$ are not $\mathcal{H}$-equivale

p144. 5– Replace "well-ordered chain" by "chain, well-ordered with respect to set inclusion,"

p146. 9+ Insert "and $\alpha < |X|$" after "σ-def $A = \{|X|\}$"